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# Heterotic Flux Tubes in $\mathcal{N} = 2$ SQCD with $\mathcal{N} = 1$ Preserving Deformations

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## Abstract

We consider non-Abelian BPS-saturated flux tubes (strings) in  $\mathcal{N} = 2$  supersymmetric QCD deformed by superpotential terms of a special type breaking  $\mathcal{N} = 2$  supersymmetry down to  $\mathcal{N} = 1$ . Previously it was believed that worldsheet supersymmetry is “accidentally” enhanced due to the facts that  $\mathcal{N} = (1, 1)$  SUSY is automatically elevated up to  $\mathcal{N} = (2, 2)$  on  $CP(N - 1)$  and, at the same time, there are no  $\mathcal{N} = (0, 2)$  generalizations of the bosonic  $CP(N - 1)$  model. Edalati and Tong noted that the target space is in fact  $CP(N - 1) \times C$  rather than  $CP(N - 1)$ . This allowed them to suggest a “heterotic”  $\mathcal{N} = (0, 2)$  sigma model, with the  $CP(N - 1)$  target space for bosonic fields and an extra right-handed fermion which couples to the fermion fields of the  $\mathcal{N} = (2, 2)$   $CP(N - 1)$  model. We derive the heterotic  $\mathcal{N} = (0, 2)$  worldsheet model directly from the bulk theory. The relation between the bulk and worldsheet deformation parameters we obtain does not coincide with that suggested by Edalati and Tong at large values of the deformation parameter. For polynomial deformation superpotentials in the bulk we find nonpolynomial response in the worldsheet model. We find a geometric representation for the heterotic model. Supersymmetry is proven to be spontaneously broken for small deformations (at the quantum level). This confirms Tong’s conjecture. A proof valid for large deformations will be presented in the subsequent publication.

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# 1 Introduction

Non-Abelian BPS-saturated flux tubes were discovered and studied in  $\mathcal{N} = 2$  supersymmetric QCD [1, 2, 3, 4, 5]. The simplest model supporting such flux tubes, to be referred to as the basic model, has the gauge group  $U(N)$ , with the  $U(1)$  Fayet–Iliopoulos (FI) term, and  $N$  flavors ( $N$  hypermultiplets in the fundamental representation). Multiple developments in supersymmetric solitons and ideas about confinement ensued (for reviews see [6, 7, 8]). A crucial feature of non-Abelian strings is the presence of orientational (and superorientational) moduli associated with rotations of their color fluxes inside a non-Abelian group, in addition to “standard” translational and supertranslational moduli. The low-energy theory on the string worldsheet is split into two disconnected parts: a free theory for (super)translational moduli and a nontrivial part, a theory of interacting (super)orientational moduli,  $CP(N - 1)$  model. The latter is completely fixed by the fact that the basic bulk theory has eight supercharges, and the string under consideration is 1/2 BPS. As well-known (e.g. [9, 10]), the only supergeneralization of the bosonic  $CP(N - 1)$  is the  $\mathcal{N} = (2, 2)$  supersymmetric  $CP(N - 1)$  model with four supercharges.

In a bid to decrease the level of supersymmetry (SUSY) in the bulk theory an  $\mathcal{N} = 2$  breaking deformation of the type

$$\mathcal{W}_{\text{deform}} = \mu \mathcal{A}^2 \tag{1.1}$$

was introduced [11] where  $\mathcal{A}$  is the adjoint chiral superfield. The above deformation preserves  $\mathcal{N} = 1$  in the bulk. As  $\mu$  increases, the adjoint fields become heavier and eventually decouple from the spectrum at  $\mu \rightarrow \infty$ .

With  $\mathcal{N} = 1$  preserving deformation of the basic model, there are four conserved supercharges in the bulk rather than eight. At the same time, the description of the orientational moduli is the same as in the  $\mathcal{N} = 2$  basic model; the bosonic part of the worldsheet theory is  $CP(N - 1)$ . Since the string solution remains 1/2 BPS, the worldsheet theory must have two conserved supercharges. Endowing the bosonic  $CP(N - 1)$  model with two supercharges automatically endows it with four supercharges [9, 10]. A conclusion was made [11] that in the problem at hand, unexpectedly, the worldsheet supersymmetry enhances up to  $\mathcal{N} = (2, 2)$ . If it were the case, the situation would be similar to supersymmetry enhancement on domain walls [12].

Recently Edalati and Tong noted [13] that the bosonic part of the worldsheet sigma model on the string is, in fact,  $CP(N - 1) \times C$  rather than  $CP(N - 1)$ , and endowing  $CP(N - 1) \times C$  with two supercharges need not necessarily lead to  $\mathcal{N} = (2, 2)$  supersymmetry on the worldsheet. They built an  $\mathcal{N} = (0, 2)$  heterotic model which supergeneralizes the bosonic model with the above target space. Moreover,

basing on a number of indirect checks they concluded that the Edalati–Tong heterotic model emerges on the string worldsheet in the  $\mathcal{N} = 1$  bulk theory and suggested a rule of converting the bulk  $\mathcal{N} = 2$  breaking superpotential into an  $\mathcal{N} = (2, 2)$  breaking superpotential on the string worldsheet.

To be more exact, the Edalati–Tong model is designed as follows. Consider for example the  $U(2)$  model in the bulk with  $CP(1)$  on the worldsheet. If  $\mathcal{N} = 2$  in the bulk is unbroken, the  $1/2$  BPS flux tube has two translational moduli associated with its center  $x_0$ , and four supertranslational moduli. The above set is totally decoupled from two orientational moduli parameterizing the coset  $SU(2)/U(1)$  accompanied by four superorientational moduli.

When  $\mathcal{N} = 2$  is broken by an  $\mathcal{N} = 1$ -preserving deformation, the number of the moduli fields remains intact, but their grouping changes. The four supertranslational moduli split into two plus two. Two left-handed fermion fields combine with  $x_0$  to form an  $\mathcal{N} = (0, 2)$  supermultiplet. These fields are described by a free theory and are decoupled from the rest of the worldsheet theory. (General aspects of two-dimensional  $\mathcal{N} = (0, 2)$  sigma models were discussed in [14].)

The right-handed fermion fields  $\zeta_R$  and  $\bar{\zeta}_R$ , which used to be “two other” supertranslational moduli, “mix” with two right-handed superorientational moduli tangential to the coset  $SU(2)/U(1)$ . Together with two orientational moduli of  $CP(1)$  and four superorientational moduli they form the  $\mathcal{N} = (0, 2)$  extension of the  $CP(1)$  model. For brevity sometimes we will refer to it as the heterotic  $CP(1)$  (or heterotic  $CP(N - 1)$  for generic  $N$ ). The fermion fields  $\zeta_R$  and  $\bar{\zeta}_R$  lie outside the target space  $SU(2)/U(1)$ . They are remnants of  $C$ . With respect to  $\mathcal{N} = (0, 2)$  supersymmetry they transform through  $F_\zeta$  terms which are expressible, via equations of motion, in terms of the fermion fields of the conventional  $CP(1)$  model.

In this paper we present a *direct derivation* of the string worldsheet theory for a generic superpotential in the bulk theory breaking  $\mathcal{N} = 2$  while preserving  $\mathcal{N} = 1$  and the  $1/2$ -BPS nature of the flux tube solution at the classical level. The  $\mu\mathcal{A}^2$  superpotential mentioned above is a particular case. Generally speaking, the minimal choice one can consider is a cubic in  $\mathcal{A}$  superpotential (in the  $U(2)$  bulk theory) with coefficients rigidly fixed by the quark mass terms. In the  $U(N)$  bulk theory with  $N_f = N$  flavors the minimal admissible  $\mathcal{N} = 1$ -preserving deformation is a polynomial of the  $(N + 1)$ -th order whose coefficients are unambiguously fixed. These more general superpotentials will be considered as well.

Focusing on the simplest example of  $U(2)$  in the bulk we prove that an  $\mathcal{N} = (0, 2)$  extension of the  $CP(1)$  model à la Edalati–Tong does indeed emerge on the string worldsheet in the low-energy limit. While gross features of the emergent heterotic worldsheet theory are those predicted by Edalati and Tong, details do not quite

coincide. In particular, for polynomial deformations in the bulk we find, generally speaking, a non-polynomial response in the worldsheet theory. Our direct derivation of the heterotic string model relies, in addition to already known results, on explicit form of the fermion zero modes on the BPS flux tubes in  $\mathcal{N} = 1$  bulk theories. To obtain the fermion zero modes we had to extend previous analyses [11, 15]. Thus, a large part of this paper bears a technical nature. It is based, however, on an observation of conceptual nature (Sect. 5) which is responsible for the very possibility of direct derivation of the heterotic  $CP(1)$  model on the string worldsheet. Indeed, in the Edalati–Tong formulation the difference between the  $\mathcal{N} = (2, 2)$  and heterotic models reveals itself in four-fermion terms. It is very hard, if possible at all, to derive these terms starting directly from the bulk theory. In our formulation the most straightforward distinction between two models occurs in the kinetic part of the Lagrangian, in the term *bilinear* in the fermion fields, of the type

$$\left(\zeta_R^\dagger \chi_R^a\right) \partial_L S^a, \quad (1.2)$$

where  $S^a$  is the bosonic field of the  $O(3)$  model subject to the constraint  $\vec{S}^2 = 1$ , while  $\chi_R^a$  is its fermionic superpartner,  $\vec{S}\vec{\chi} = 0$ . Since the term in (1.2) is bilinear in the fermion fields, the knowledge of the fermion zero modes allows one to get this term from the bulk Lagrangian in a very explicit and direct way. Other additional terms transforming  $\mathcal{N} = (2, 2)$  model into  $\mathcal{N} = (0, 2)$  unambiguously follow from (1.2) by virtue of  $\mathcal{N} = (0, 2)$  supersymmetry.

The basic features of the heterotic  $CP(1)$  model we obtain are as follows. The term (1.2) entails the occurrence of the four-fermion interaction of the type

$$\left(\zeta_R^\dagger \zeta_R\right) \left(\chi_L^a \chi_L^b\right) S^c \varepsilon_{abc}, \quad (1.3)$$

and a suppression of the coefficient in front of the conventional four-fermion term

$$\frac{1}{2} \left(\chi_L^a \chi_R^a\right)^2. \quad (1.4)$$

The addition of seemingly rather insignificant  $\zeta_R, \bar{\zeta}_R$  terms to the  $\mathcal{N} = (2, 2)$   $CP(N-1)$  model drastically changes its dynamical behavior. In particular, Witten’s index  $I = N$  for  $CP(N-1)$  [16] changes and becomes zero. Supersymmetry on the worldsheet is no longer protected by Witten’s index. In fact, we will prove, at small  $\mu$ , that spontaneous SUSY breaking does take place. The fields  $\zeta_R, \zeta_R^\dagger$  play the role of Goldstinos. In the accompanying paper [17] we will solve the heterotic  $CP(N-1)$  model at large  $N$  and prove that supersymmetry is spontaneously broken at the quantum level for any value of the deformation parameter, as was anticipated

by Tong [18]. This result seems to be intuitively clear given that small variations of the deformation superpotential ruin the BPS nature of the flux-tube solutions already at the classical level.

We will derive a long-sought geometric representation of the heterotic  $\mathcal{N} = (0, 2)$  model, in terms of the metric and curvature tensor of the  $CP(N - 1)$  space.

Organization of the paper is as follows. In Sect. 2 we review our basic bulk theory with eight supercharges and discuss possible deformations of this bulk theory breaking  $\mathcal{N} = 2$  down to  $\mathcal{N} = 1$  without destroying the BPS nature of the flux-tube solution. In Sect. 3 we review construction of non-Abelian strings in the  $\mathcal{N} = (2, 2)$  limit. Moreover, we perform derivation of those fermion zero modes which had not been explicitly derived in the literature previously. Section 4 summarizes general aspects of the Edalati–Tong model. In Sect. 5 we present our formulation of the heterotic  $CP(1)$  model. Section 6 is devoted to yet another, geometric, formulation of the heterotic  $CP(1)$  model. Here we also show that at small  $\mu$  the vacuum energy density of the heterotic model is proportional to the square of the chiral condensate. In Sect. 7 we begin our direct derivation of the worldsheet model from the bulk theory deformed by the superpotential (1.1). Section 7 is devoted to the fermion zero modes. Section 8 establishes the relation between the parameters of the worldsheet model and those of the bulk theory. In Sects. 9 and 10 we proceed to a more general case of a polynomial deformation superpotential replacing the simplest superpotential (1.1). Here we calculate the worldsheet superpotential in two limits,  $\mu \rightarrow 0$  and  $\mu \rightarrow \infty$ . While the first result agrees with the Edalati–Tong conjecture, the large- $\mu$  limit defies it. We show that in this case the main effect of  $\mathcal{N} = 2$  breaking deformation in the  $\mu \rightarrow \infty$  limit is that the potential of the worldsheet theory gets enhanced. It forces the string orientational vector to point towards the north or south poles of the sphere  $S_2 = SU(2)/U(1)$ . The string becomes exceedingly more “Abelian” as we increase the deformation superpotential in the bulk. Section 11 summarizes our findings.

**Remark:** In Sects. 2–5 and 7–10 we use Euclidean notation most suitable for consideration of static solitons. This is explained in Appendix A. Section 6 which bears a general nature is presented in Minkowski notation. This is explained in Appendix B. In Appendix C we briefly discuss the Witten index for the heterotic  $\mathcal{N} = (0, 2)$   $CP(N - 1)$  models. In Appendix D we collect for convenience various definitions of the deformation parameters.

## 2 Bulk theory

The gauge symmetry of the basic bulk model is  $SU(N) \times U(1)$ . We will focus on the  $SU(2) \times U(1)$  case, which presents the simplest example. Besides the gauge bosons, gauginos and their superpartners, the model has the matter sector consisting of  $N_f = N = 2$  “quark” hypermultiplets. In addition, we will introduce the Fayet–Iliopoulos  $D$ -term for the  $U(1)$  gauge field which triggers the quark condensation.

Let us first discuss the undeformed theory with  $\mathcal{N} = 2$ . The superpotential has the form

$$\mathcal{W}_{\mathcal{N}=2} = \frac{1}{\sqrt{2}} \sum_{A=1}^2 \left( \tilde{q}_A \mathcal{A} q^A + \tilde{q}_A \mathcal{A}^a \tau^a q^A \right), \quad (2.1)$$

where  $\mathcal{A}$  and  $\mathcal{A}^a$  are chiral superfields, the  $\mathcal{N} = 2$  superpartners of the gauge bosons of  $U(1)$  and  $SU(2)$ , respectively. Furthermore,  $q_A$  and  $\tilde{q}_A$  ( $A = 1, 2$ ) represent two matter (quark) hypermultiplets. The flavor index is denoted by  $A$ . Thus, in our model the number of colors coincides with the number of flavors. The  $q^A$  mass terms are denoted by  $m_A$ .

Next, we add a superpotential which breaks supersymmetry down to  $\mathcal{N} = 1$ . In this paper we will consider two types of  $\mathcal{N} = 1$  preserving deformation superpotentials. The first superpotential is the mass term for the adjoint fields,

$$\mathcal{W}_{3+1} = \frac{\mu}{2} \left[ \mathcal{A}^2 + (\mathcal{A}^a)^2 \right], \quad (2.2)$$

where  $\mu$  is a common mass parameter for the chiral superfields in  $\mathcal{N} = 2$  gauge supermultiplets,  $U(1)$  and  $SU(2)$ , respectively. The subscript 3+1 tells us that the deformation superpotential (2.2) refers to the bulk four-dimensional theory. Clearly, the mass term (2.2) splits  $\mathcal{N} = 2$  supermultiplets, breaking  $\mathcal{N} = 2$  supersymmetry down to  $\mathcal{N} = 1$ .

For the deformation (2.2), in order to preserve the BPS nature of the flux-tube solutions, it is necessary to set the quark mass terms at zero,

$$m_1 = m_2 = 0. \quad (2.3)$$

As was shown in [11] and [13] (see also the review paper [8]), in this case the deformed theory supports 1/2 BPS -saturated flux-tube solutions at the classical level.

The second (more general) deformation we will consider in this paper is a polynomial superpotential of the form

$$\mathcal{W}_{3+1} = \text{Tr} \sum_{k=1}^{N=2} \frac{c_k}{k+1} \hat{\mathcal{A}}^{k+1}, \quad (2.4)$$

where we introduce the adjoint matrix superfield

$$\hat{\mathcal{A}} = \frac{1}{2} \mathcal{A} + \frac{\tau^a}{2} \mathcal{A}^a; \quad (2.5)$$

$\tau^a$  are the SU(2) Pauli matrices. The hat over  $\mathcal{A}$  will remind us that  $\mathcal{A}$  is a matrix from U(2) rather than SU(2). The coefficients  $c_k$  are not arbitrary. As explained at the end of this section, they are unambiguously fixed by the bulk theory parameters.

The bosonic part of our SU(2)×U(1) theory has the form

$$\begin{aligned} S = & \int d^4x \left[ \frac{1}{4g_2^2} (F_{\mu\nu}^a)^2 + \frac{1}{4g_1^2} (F_{\mu\nu})^2 + \frac{1}{g_2^2} |D_\mu a^a|^2 + \frac{1}{g_1^2} |\partial_\mu a|^2 \right. \\ & \left. + |\nabla_\mu q^A|^2 + |\nabla_\mu \tilde{q}^A|^2 + V(q^A, \tilde{q}_A, a^a, a) \right]. \end{aligned} \quad (2.6)$$

Here  $D_\mu$  is the covariant derivative in the adjoint representation of SU(2), while

$$\nabla_\mu = \partial_\mu - \frac{i}{2} A_\mu - i A_\mu^a \frac{\tau^a}{2}. \quad (2.7)$$

The coupling constants  $g_1$  and  $g_2$  correspond to the U(1) and SU(2) sectors respectively. With our conventions, the U(1) charges of the fundamental matter fields are  $\pm 1/2$ .

The potential  $V(q^A, \tilde{q}_A, a^a, a)$  in the Lagrangian (2.6) is a sum of various  $D$  and  $F$  terms,

$$\begin{aligned} V(q^A, \tilde{q}_A, a^a, a) = & \frac{g_2^2}{2} \left( \frac{1}{g_2^2} \varepsilon^{abc} \tilde{a}^b a^c + \tilde{q}_A \frac{\tau^a}{2} q^A - \tilde{q}_A \frac{\tau^a}{2} \tilde{q}^A \right)^2 \\ & + \frac{g_1^2}{8} (\tilde{q}_A q^A - \tilde{q}_A \tilde{q}^A - 2\xi)^2 \\ & + \frac{g_2^2}{2} \left| \tilde{q}_A \tau^a q^A + \sqrt{2} \frac{\partial \mathcal{W}_{3+1}}{\partial a^a} \right|^2 + \frac{g_1^2}{2} \left| \tilde{q}_A q^A + \sqrt{2} \frac{\partial \mathcal{W}_{3+1}}{\partial a} \right|^2 \\ & + \frac{1}{2} \sum_{A=1}^2 \left\{ \left| (a + \tau^a a^a + \sqrt{2} m^A) q^A \right|^2 \right. \\ & \left. + \left| (a + \tau^a a^a + \sqrt{2} m^A) \tilde{q}_A \right|^2 \right\}, \end{aligned} \quad (2.8)$$

where the sum over repeated flavor indices  $A$  is implied. The first and second lines here represent  $D$  terms, the third line the  $F_A$  terms, while the fourth and the fifth lines represent the squark  $F$  terms. We also introduced the Fayet–Iliopoulos  $D$ -term



for the U(1) field, with the FI parameter  $\xi$  in (2.8). Note that the Fayet–Iliopoulos term does not break  $\mathcal{N} = 2$  supersymmetry [19, 15]. The parameters which do break  $\mathcal{N} = 2$  down to  $\mathcal{N} = 1$  are  $\mu$  or  $c_k$  in (2.2) or (2.4).

The vacuum structure and the mass spectrum of perturbative excitations in this theory were studied in [11] for the case of mass-type deformation (2.2). Here we briefly review relevant results for convenience.

The Fayet–Iliopoulos term triggers the spontaneous breaking of the gauge symmetry. The vacuum expectation values (VEV’s) of the squark fields can be chosen as

$$\begin{aligned} \langle q^{kA} \rangle &= \sqrt{\xi} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \langle \bar{q}^{kA} \rangle = 0, \\ k &= 1, 2, \quad A = 1, 2, \end{aligned} \tag{2.9}$$

while the VEV’s of the adjoint fields vanish

$$\langle a^a \rangle = 0, \quad \langle a \rangle = 0. \tag{2.10}$$

Here we write down  $q$  as a  $2 \times 2$  matrix, the first superscript ( $k = 1, 2$ ) refers to SU(2) color, while the second ( $A = 1, 2$ ) to flavor. We keep the quark masses  $m_1 = m_2 = 0$  in conjunction with (2.2).

The color-flavor locked form of the quark VEV’s in Eq. (2.9) and the absence of VEV of the adjoint scalar  $a^a$  in Eq. (2.10) results in the fact that, while the theory is fully Higgsed, a diagonal SU(2)<sub>C+F</sub> survives as a global symmetry. The presence of this symmetry leads to the emergence of orientational zero modes of  $Z_2$  strings in the model (2.6) [2].

With two matter hypermultiplets, the SU(2) part of the gauge group is asymptotically free, implying generation of a dynamical scale  $\Lambda$ . In order to stay at weak coupling we assume that  $\sqrt{\xi} \gg \Lambda$ , so that the SU(2) coupling running is frozen by the squark condensation at a small value.

Since both U(1) and SU(2) gauge groups are broken by the squark condensation, all gauge bosons become massive. From (2.6) we get for the U(1) gauge boson

$$m_\gamma = g_1 \sqrt{\xi}, \tag{2.11}$$

while three gauge bosons of the SU(2) group acquire the same mass

$$m_W = g_2 \sqrt{\xi}. \tag{2.12}$$

To get the masses of the scalar bosons we expand the potential (2.8) near the vacuum (2.9), (2.10) and diagonalize the corresponding mass matrix. The four components of the eight-component<sup>1</sup> scalar  $q^{kA}$  are eaten by the Higgs mechanism for

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<sup>1</sup>We mean here eight *real* components.

U(1) and SU(2) gauge groups. Another four components are split as follows: one component acquires the mass (2.11). It becomes a scalar component of a massive  $\mathcal{N} = 1$  vector U(1) gauge multiplet. Other three components acquire masses (2.12) and become scalar superpartners of the SU(2) gauge boson in  $\mathcal{N} = 1$  massive gauge supermultiplet.

Other 16 real scalar components of the fields  $\tilde{q}_{Ak}$ ,  $a^a$  and  $a$  produce the following states: two states acquire mass

$$m_{\text{U}(1)}^+ = g_1 \sqrt{\xi \lambda_1^+}, \quad (2.13)$$

while the mass of other two states is given by

$$m_{\text{U}(1)}^- = g_1 \sqrt{\xi \lambda_1^-}, \quad (2.14)$$

where  $\lambda_1^\pm$  are two roots of the quadratic equation

$$\lambda_i^2 - \lambda_i(2 + \omega_i^2) + 1 = 0, \quad (2.15)$$

for  $i = 1$ . Here we introduced two  $\mathcal{N} = 2$  supersymmetry breaking parameters associated with the U(1) and SU(2) gauge groups, respectively,

$$\omega_1 = \frac{g_1^2 \mu}{m_\gamma}, \quad \omega_2 = \frac{g_2^2 \mu}{m_W}. \quad (2.16)$$

Furthermore, other  $2 \times 3 = 6$  states acquire mass

$$m_{\text{SU}(2)}^+ = g_2 \sqrt{\xi \lambda_2^+}, \quad (2.17)$$

while the remaining  $2 \times 3 = 6$  states also become massive. Their mass is

$$m_{\text{SU}(2)}^- = g_2 \sqrt{\xi \lambda_2^-}. \quad (2.18)$$

Here  $\lambda_2^\pm$  are two roots of the quadratic equation (2.15) for  $i = 2$ . Note that all states come either as singlets or triplets of unbroken  $\text{SU}(2)_{C+F}$ .

In the large- $\mu$  limit the larger masses  $m_{\text{U}(1)}^+$  and  $m_{\text{SU}(2)}^+$  become

$$m_{\text{U}(1)}^+ = m_{\text{U}(1)} \omega_1 = g_1^2 \mu, \quad m_{\text{SU}(2)}^+ = m_{\text{SU}(2)} \omega_2 = g_2^2 \mu. \quad (2.19)$$

Clearly, in the limit  $\mu \rightarrow \infty$  these are the masses of the heavy adjoint scalars  $a$  and  $a^a$ . At  $\omega_i \gg 1$  these fields decouple and can be integrated out.

The low-energy bulk theory in this limit contains massive gauge  $\mathcal{N} = 1$  multiplets and chiral multiplets with lower masses  $m_{\text{U}(1), \text{SU}(2)}^-$ . Equation (2.15) gives for these masses

$$m_{\text{U}(1)}^- = \frac{m_{\text{U}(1)}}{\omega_1} = \frac{\xi}{\mu}, \quad m_{\text{SU}(2)}^- = \frac{m_{\text{SU}(2)}}{\omega_2} = \frac{\xi}{\mu}. \quad (2.20)$$

In the limit of infinite  $\mu$  these masses tend to zero. This fact reflects the emergence of a Higgs branch in  $\mathcal{N} = 1$  SQCD, see, for example, [20].

As was explained in [11], the presence of the Higgs branch in the  $\mu \rightarrow \infty$  limit is quite an unpleasant feature of the theory (2.6). The presence of quark massless states in the bulk associated with this Higgs branch obscure physics of the non-Abelian strings in this theory. In particular, the strings become infinitely thick. This means that higher derivative corrections in the effective theory on the string become important. In [11] the maximal critical value of the parameter  $\mu$  was estimated beyond which one can no longer trust the effective low-energy theory on the string worldsheet,

$$g_2^2 \mu \ll \frac{m_W^3}{\Lambda_{\mathcal{N}=1}^2}, \quad (2.21)$$

where  $\Lambda_{\mathcal{N}=1}$  is the scale of  $\mathcal{N} = 1$  SQCD to which the theory (2.6) flows in the large- $\mu$  limit,

$$\Lambda_{\mathcal{N}=1}^4 = g_2^4 \mu^2 \Lambda^2. \quad (2.22)$$

We assume that the condition (2.21) is met.

We still have a large window for the values of the  $\mu$  parameter, with  $\mu$  staying below the upper bound (2.21), but, on the other hand, large enough to ensure the decoupling of the adjoint fields, namely<sup>2</sup>

$$m_W \ll g_2^2 \mu \ll m_W \frac{m_W^2}{\Lambda_{\mathcal{N}=1}^2}. \quad (2.23)$$

To conclude this section we briefly discuss a more general deformation of  $\mathcal{N} = 2$  SQCD given by the superpotential (2.4). As was shown in [13], in order to preserve the BPS nature of the string solutions, one has to consider a deformation superpotential (2.4) of a special type, with the critical points coinciding with the quark mass terms. In the U(2) case this boils down to

$$\frac{\partial \mathcal{W}_{3+1}}{\partial \hat{\mathcal{A}}} = \text{Tr} \sum_{k=1}^{N=2} c_k \hat{\mathcal{A}}^k = \frac{\mu}{\Delta m} \text{Tr} \left( \hat{\mathcal{A}} + \frac{m_1}{\sqrt{2}} \right) \left( \hat{\mathcal{A}} + \frac{m_2}{\sqrt{2}} \right), \quad (2.24)$$

where  $\mu$  is the deformation parameter and

$$\Delta m = m_1 - m_2. \quad (2.25)$$

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<sup>2</sup>When we speak of sending  $\mu$  to  $\infty$  we in fact mean that  $\mu$  lies near the upper edge of the window (2.23). The dimensionless parameter determining whether  $\mu$  is small or large is  $g_2^2 \mu / m_W$ . When  $\mu$  is close to the upper edge of the window (2.23) for all practical purposes we can put the above parameter to  $\infty$ .

With this superpotential added, the squark VEV's are given by the same expression (2.9) as for the adjoint mass deformation (2.2), while the adjoint VEV's are now

$$\hat{a} = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}. \quad (2.26)$$

It is rather obvious that deviations of the coefficients  $c_k$  from (2.24) eliminate BPS saturated flux-tube solutions, see Sect. 3.

The deformation (2.24) in the large- $\mu$  limit gives large masses, of the order of  $g^2\mu$ , to the adjoint fields  $a$  and  $a^3$  leaving the fields  $a^{1,2}$  intact (with masses of the order of  $g\sqrt{\xi}$ , we assume that  $\Delta m \ll g\sqrt{\xi}$ ). Thus, in the large- $\mu$  limit the breaking of the  $U(2)$  gauge group by the adjoint VEV's is not washed out. Instead, it becomes stronger as we increase  $\mu$ . The theory with the deformation (2.24) does not flow to  $\mathcal{N} = 1$  SQCD at  $\mu \rightarrow \infty$  because  $m(a^{1,2})$  stays frozen at  $g\sqrt{\xi}$ . In this sense, the mass-type deformation (2.2) is more efficient.

Below in Sects. 3–8 we discuss the deformation of  $\mathcal{N} = 2$  SQCD with the monomial superpotential (2.2), and then in Sects. 9 and 10 consider the polynomial deformation superpotential (2.24).

### 3 Non-Abelian strings

Non-Abelian strings were shown to emerge at weak coupling in  $\mathcal{N} = 2$  supersymmetric gauge theories with the  $U(N)$  gauge group [1, 2, 3, 5], see also the review papers [6, 8]. The main feature of the non-Abelian strings is the presence of orientational zero modes associated with rotations of their color flux in the non-Abelian gauge group, which makes such strings genuinely non-Abelian. This solution of the  $\mathcal{N} = 2$  theory was generalized to the theory with the mass term deformation (2.2) in [11]. The reason why the string solution remains BPS-saturated (at the classical level) after the deformation (2.2) is switched on is as follows: classically the flux-tube solution is constructed from the gauge and  $q$  fields which have the same masses, see Eqs. (2.12) and (2.11). The fields  $\tilde{q}$  and  $a$  are given by their vanishing VEV's, see (2.9) and (2.10). If we considered more generic deformations (say, a deformation of the type (2.4) with polynomial superpotentials) the fields  $\tilde{q}$  and  $a$  would be excited in the flux-tube solution implying the loss of the BPS saturation. The reason is that the fields  $\tilde{q}$  and  $a$  have masses different from those of the gauge bosons and  $q$  fields.

Below in this and subsequent sections we will consider the mass term deformation (2.2) which does not excite the fields  $\tilde{q}$ , and  $a$ , and the string remains classically BPS-saturated.<sup>3</sup>

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<sup>3</sup>The  $\tilde{q}$  and  $a$  fields are, of course, present at the quantum level. This raises the issue of a possible

The  $Z_2$  string solution (a progenitor of the non-Abelian string) can be written as follows [2]:

$$\begin{aligned} q(x) &= \begin{pmatrix} e^{i\alpha}\phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix}, \\ A_i^3(x) &= -\varepsilon_{ij} \frac{x_j}{r^2} (1 - f_3(r)), \\ A_i(x) &= -\varepsilon_{ij} \frac{x_j}{r^2} (1 - f(r)), \end{aligned} \tag{3.1}$$

where  $i = 1, 2$  labels coordinates in the plane orthogonal to the string axis and  $r$  and  $\alpha$  are the polar coordinates in this plane. The profile functions  $\phi_1(r)$  and  $\phi_2(r)$  determine the profiles of the scalar fields, while  $f_3(r)$  and  $f(r)$  determine the SU(2) and U(1) gauge fields of the string solution, respectively. These functions satisfy the following first-order equations [2]:

$$\begin{aligned} r \frac{d}{dr} \phi_1(r) - \frac{1}{2} (f(r) + f_3(r)) \phi_1(r) &= 0, \\ r \frac{d}{dr} \phi_2(r) - \frac{1}{2} (f(r) - f_3(r)) \phi_2(r) &= 0, \\ -\frac{1}{r} \frac{d}{dr} f(r) + \frac{g_1^2}{2} [(\phi_1(r))^2 + (\phi_2(r))^2 - 2\xi] &= 0, \\ -\frac{1}{r} \frac{d}{dr} f_3(r) + \frac{g_2^2}{2} [(\phi_1(r))^2 - (\phi_2(r))^2] &= 0. \end{aligned} \tag{3.2}$$

The boundary conditions for the profile functions in these equations are

$$\begin{aligned} f_3(0) &= 1, & f(0) &= 1; \\ f_3(\infty) &= 0, & f(\infty) &= 0 \end{aligned} \tag{3.3}$$

for the gauge fields, while the boundary conditions for the squark fields are

$$\phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}, \quad \phi_1(0) = 0. \tag{3.4}$$

Note that since the field  $\phi_2$  does not wind, it need not vanish at the origin, and, in fact, it does not. Numerical solutions of the Bogomol'nyi equations (3.2) for the  $Z_2$  strings were found in Ref. [2].

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breaking of the string “BPSness” at the quantum level (a spontaneous supersymmetry breaking in the effective theory on the string worldsheet). In [18] it was argued that  $\mathcal{N} = (0, 2)$  supersymmetry in the two-dimensional theory on the string worldsheet is spontaneously broken, indeed. This argument is confirmed in Sect. 6 and by the exact solution at large  $N$ , see [17].

The tension of the elementary  $Z_2$  string is

$$T = 2\pi \xi, \quad (3.5)$$

to be compared with the tension of the Abelian Abrikosov–Nielsen–Olesen (ANO) string [21],

$$T_{\text{ANO}} = 4\pi \xi \quad (3.6)$$

in our normalization.

Making the elementary  $Z_2$  strings “wind” in  $\text{SU}(2)$  makes it *bona fide* non-Abelian. This means that, besides trivial translational moduli, the string acquires  $\text{SU}(2)/\text{U}(1)$  moduli. Indeed, while the “flat” vacuum (2.9) is  $\text{SU}(2)_{C+F}$  symmetric, the solution (3.1) breaks this symmetry down to  $\text{U}(1)$  which gives rise to a family of degenerate solutions.

To obtain the above family from the  $Z_2$  string (3.1) we act on it by diagonal color-flavor rotations preserving the vacuum (2.9). To this end it is convenient to pass to the singular gauge where the scalar fields have no winding at infinity, while the string flux comes from the vicinity of the origin. In this gauge we have

$$q = U \begin{pmatrix} \phi_1(r) & 0 \\ 0 & \phi_2(r) \end{pmatrix} U^{-1} = \frac{1}{2}(\phi_1 + \phi_2) + \frac{\tau^a}{2} S^a (\phi_1 - \phi_2),$$

$$A_i^a(x) = S^a \varepsilon_{ij} \frac{x_j}{r^2} f_3(r), \quad A_i(x) = \varepsilon_{ij} \frac{x_j}{r^2} f(r), \quad (3.7)$$

where  $U$  is a matrix  $\in \text{SU}(2)$  and  $S^a$  is a moduli vector defined as

$$S^a \tau^a = U \tau^3 U^{-1}, \quad a = 1, 2, 3, \quad (3.8)$$

and subject to the constraint

$$\vec{S}^2 = 1. \quad (3.9)$$

At  $S = \{0, 0, 1\}$  we get the field configuration quoted in Eq. (3.1).

As soon as the  $\text{SU}(2)_{C+F}$  group is broken by the string solution (3.1) down to  $\text{U}(1)$ , the effective two-dimensional theory on the string which describes the internal dynamics of the orientational moduli  $S^a$  is the  $O(3) = CP(1)$  model ( $CP(N-1)$  in the general case of the  $\text{SU}(N) \times \text{U}(1)$  gauge group in the bulk theory) [1, 2, 3, 5, 22]. The bosonic action has the form (for derivation see the review paper [8])

$$S^{(1+1)} = \frac{\beta}{2} \int dt dz (\partial_k S^a)^2, \quad (3.10)$$

where the coupling constant  $\beta$  is given by a normalizing integral

$$\beta = \frac{2\pi}{g_2^2} \int_0^\infty dr \left\{ -\frac{d}{dr} f_3 + \left( \frac{2}{r} f_3^2 + \frac{d}{dr} f_3 \right) \frac{\phi_1^2}{\phi_2^2} \right\}. \quad (3.11)$$

Using the first-order equations for the string profile functions (3.2) one can see that the integral here reduces to a total derivative and is given by the flux of the string determined by  $f_3(0) = 1$ . Thus

$$\beta = \frac{2\pi}{g_2^2}. \quad (3.12)$$

The two-dimensional coupling constant is determined by the four-dimensional non-Abelian coupling.

The above relation between the four-dimensional and two-dimensional coupling constants (3.12) is obtained at the classical level. In quantum theory both couplings run. In particular, the  $CP(1)$  model is asymptotically free [23] and develops its own scale  $\Lambda_{CP(1)}$ . Its relation to the parameters of the bulk theory (2.6) is given by

$$\Lambda_{CP(1)} = \frac{\Lambda_{\mathcal{N}=1}^2}{m_W}, \quad (3.13)$$

see Ref. [11].

### 3.1 Fermion zero modes: $\mathcal{N} = 2$ limit

Let us start from the  $\mathcal{N} = 2$  theory (2.6) with no deformation superpotential. Our string solution is 1/2 BPS-saturated. This means that four supercharges, out of eight of the four-dimensional theory (2.6), act trivially on the string solution (3.7). The remaining four supercharges generate four fermion zero modes which we call supertranslational modes because they are superpartners to two translational zero modes. The corresponding four fermionic moduli are superpartners to the coordinates  $x_0$  and  $y_0$  of the string center. The supertranslational fermion zero modes were found in Ref. [15] for the Abelian ANO string. Below we generalize this construction to the case of the non-Abelian string.

The fermionic part of the action of the model (2.6) is

$$\begin{aligned} S_{\text{ferm}} = & \int d^4x \left\{ \frac{i}{g_2^2} \bar{\lambda}_f^a \bar{D} \lambda^{af} + \frac{i}{g_1^2} \bar{\lambda}_f \bar{D} \lambda^f + \text{Tr} [\bar{\psi} i \bar{\nabla} \psi] + \text{Tr} [\tilde{\psi} i \tilde{\nabla} \tilde{\psi}] \right. \\ & + \frac{i}{\sqrt{2}} \text{Tr} [\bar{q}_f (\lambda^f \psi) + (\tilde{\psi} \lambda_f) q^f + (\bar{\psi} \bar{\lambda}_f) q^f + \bar{q}^f (\bar{\lambda}_f \tilde{\psi})] \\ & + \frac{i}{\sqrt{2}} \text{Tr} [\bar{q}_f \tau^a (\lambda^{af} \psi) + (\tilde{\psi} \lambda_f^a) \tau^a q^f + (\bar{\psi} \bar{\lambda}_f^a) \tau^a q^f + \bar{q}^f \tau^a (\bar{\lambda}_f^a \tilde{\psi})] \\ & + \frac{i}{\sqrt{2}} \text{Tr} [\tilde{\psi} (a + a^a \tau^a) \psi] + \frac{i}{\sqrt{2}} \text{Tr} [\bar{\psi} (a + a^a \tau^a) \tilde{\psi}] \\ & \left. - \frac{\mu}{2} (\lambda^2)^2 - \frac{\mu}{2} (\lambda^{a2})^2 \right\}, \quad (3.14) \end{aligned}$$

where the matrix color-flavor notation is used for the matter fermions  $(\psi^\alpha)^{kA}$  and  $(\tilde{\psi}^\alpha)_{Ak}$ . The traces are performed over the color-flavor indices. Contraction of the spinor indices is assumed inside all parentheses, for instance,  $(\lambda\psi) \equiv \lambda_\alpha\psi^\alpha$ , see Appendix A. We write the squark fields in (3.14) as doublets of the  $SU(2)_R$  group which is present in  $\mathcal{N} = 2$  theory,  $q^f = (q, \bar{q})$ . Here  $f = 1, 2$  is the  $SU(2)_R$  index which labels two supersymmetries of the bulk theory in the  $\mathcal{N} = 2$  limit. Moreover,  $\lambda^{\alpha f}$  and  $(\lambda^{\alpha f})^a$  stand for the gauginos of the  $U(1)$  and  $SU(2)$  groups, respectively. Note that the last two terms are  $\mathcal{N} = 1$  deformations in the fermion sector of the theory induced by the breaking parameter  $\mu$ . They involve only  $f = 2$  components of  $\lambda$ 's explicitly breaking the  $SU(2)_R$  invariance.

Now, we put  $\mu = 0$  (consideration of  $\mu \neq 0$  will be carried out in Sect. 7) and apply supersymmetry transformations to generate four supertranslational modes of the non-Abelian string in the  $\mathcal{N} = 2$  limit. The supertransformations in our bulk theory have the form

$$\begin{aligned}\delta\lambda^{f\alpha} &= \frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu\epsilon^f)^\alpha F_{\mu\nu} + \epsilon^{\alpha p}D^m(\tau^m)_p^f + \dots, \\ \delta\lambda^{af\alpha} &= \frac{1}{2}(\sigma_\mu\bar{\sigma}_\nu\epsilon^f)^\alpha F_{\mu\nu}^a + \epsilon^{\alpha p}D^{am}(\tau^m)_p^f + \dots, \\ \delta\bar{\psi}_{\dot{\alpha}}^{kA} &= i\sqrt{2}\bar{\nabla}_{\dot{\alpha}\alpha}q_f^{kA}\epsilon^{\alpha f} + \dots, \\ \delta\bar{\psi}_{\dot{\alpha}Ak} &= i\sqrt{2}\bar{\nabla}_{\dot{\alpha}\alpha}\bar{q}_{fAk}\epsilon^{\alpha f} + \dots.\end{aligned}\tag{3.15}$$

Here the parameters of SUSY transformations are denoted as  $\epsilon^{\alpha f}$ . Furthermore, the  $D$  terms in Eq. (3.15) are

$$D^1 + iD^2 = 0, \quad D^3 = -i\frac{g_1^2}{2}(\text{Tr}|q|^2 - 2\xi)\tag{3.16}$$

for the  $U(1)$  field, and

$$D^{a1} + iD^{a2} = 0, \quad D^{a3} = -i\frac{g_2^2}{2}\text{Tr}(\bar{q}\tau^a q)\tag{3.17}$$

for the  $SU(2)$  field. The dots in (3.15) stand for terms involving the adjoint scalar fields which vanish on the string solution (at  $m_1 = m_2$ ) because the adjoint fields are given by their vacuum expectation values (2.10).

In Ref. [15] it was shown that the four supercharges associated with the parameters  $\epsilon^{12}$  and  $\epsilon^{21}$  act trivially on the BPS string in the theory with the Fayet–Iliopoulos  $D$  term. The same is true for the non-Abelian string solution (3.7). Applying supertransformations (3.15) with the parameters  $\epsilon^{11}$  and  $\epsilon^{22}$  to (3.7) we generate the



following supertranslational zero modes:

$$\begin{aligned}
\bar{\psi}_{Ak\dot{2}} &= -\frac{1}{\sqrt{2}} \frac{x_1 + ix_2}{r^2} \{[(f + f_3)\phi_1 + (f - f_3)\phi_2] \\
&\quad + \tau^a S^a [(f + f_3)\phi_1 - (f - f_3)\phi_2]\} \zeta_L, \\
\bar{\psi}_1^{kA} &= \frac{1}{\sqrt{2}} \frac{x_1 - ix_2}{r^2} \{[(f + f_3)\phi_1 + (f - f_3)\phi_2] \\
&\quad + \tau^a S^a [(f + f_3)\phi_1 - (f - f_3)\phi_2]\} \zeta_R, \\
\bar{\psi}_{Ak\dot{1}} &= 0, \quad \bar{\psi}_2^{kA} = 0, \\
\lambda^{a22} &= ig_2^2 (\phi_1^2 - \phi_2^2) S^a \zeta_R, \\
\lambda^{a11} &= -ig_2^2 (\phi_1^2 - \phi_2^2) S^a \zeta_L, \\
\lambda^{a12} &= 0, \quad \lambda^{a21} = 0, \\
\lambda^{22} &= ig_1^2 (\phi_1^2 + \phi_2^2 - 2\xi) \zeta_R, \\
\lambda^{11} &= -ig_1^2 (\phi_1^2 + \phi_2^2 - 2\xi) \zeta_L, \\
\lambda^{12} &= 0, \quad \lambda^{21} = 0,
\end{aligned} \tag{3.18}$$

where the dependence on  $x_i$  is encoded in the string profile functions, see Eq. (3.7), while the Grassmann parameters  $\zeta_L$  and  $\zeta_R$  are related to the SUSY transformation parameters,

$$\delta\zeta_L = \epsilon^{11}, \quad \delta\zeta_R = \epsilon^{22}. \tag{3.19}$$

These parameters become superpartners of the string center coordinates  $x_i$  ( $i = 1, 2$ ) in the effective theory on the string worldsheet.

Besides four supertranslational modes the non-Abelian string has four superorientational modes. They were calculated in [3] using supersymmetry transformations (3.15) with the parameters  $\epsilon^{12}$  and  $\epsilon^{21}$ . They have the following form:

$$\begin{aligned}
\bar{\psi}_{Ak\dot{2}} &= \left(\frac{\tau^a}{2}\right)_{Ak} \frac{1}{2\phi_2} (\phi_1^2 - \phi_2^2) [\chi_L^a + i\varepsilon^{abc} S^b \chi_L^c], \\
\bar{\psi}_1^{kA} &= \left(\frac{\tau^a}{2}\right)^{kA} \frac{1}{2\phi_2} (\phi_1^2 - \phi_2^2) [\chi_R^a - i\varepsilon^{abc} S^b \chi_R^c],
\end{aligned}$$

$$\begin{aligned}
\bar{\psi}_{Aki} &= 0, & \bar{\psi}_2^{kA} &= 0, \\
\lambda^{a22} &= \frac{i}{\sqrt{2}} \frac{x_1 + ix_2}{r^2} f_3 \frac{\phi_1}{\phi_2} \left[ \chi_R^a - i\varepsilon^{abc} S^b \chi_R^c \right], \\
\lambda^{a11} &= \frac{i}{\sqrt{2}} \frac{x_1 - ix_2}{r^2} f_3 \frac{\phi_1}{\phi_2} \left[ \chi_L^a + i\varepsilon^{abc} S^b \chi_L^c \right], \\
\lambda^{a12} &= 0, & \lambda^{a21} &= 0,
\end{aligned} \tag{3.20}$$

where  $\chi_L^a$  and  $\chi_R^a$  are real Grassmann parameters, subject to constraints

$$S^a \chi_L^a = 0, \quad S^a \chi_R^a = 0. \tag{3.21}$$

We can directly verify that the zero modes (3.18) and (3.20) satisfy the Dirac equations of motion. From the fermion action of the model (3.14) we get the relevant Dirac equations for  $\lambda^a$ ,

$$\begin{aligned}
\frac{i}{g_1^2} \bar{D} \lambda^f + \frac{i}{\sqrt{2}} \text{Tr} \left( \bar{\psi} q^f + \bar{q}^f \tilde{\psi} \right) - \mu \delta_2^f \bar{\lambda}_2 &= 0, \\
\frac{i}{g_2^2} \bar{D} \lambda^{af} + \frac{i}{\sqrt{2}} \text{Tr} \left( \bar{\psi} \tau^a q^f + \bar{q}^f \tau^a \tilde{\psi} \right) - \mu \delta_2^f \bar{\lambda}_2^a &= 0,
\end{aligned} \tag{3.22}$$

while for the matter fermions

$$\begin{aligned}
i \nabla \bar{\psi} + \frac{i}{\sqrt{2}} \left[ \bar{q}_f \lambda^f - (\tau^a \bar{q}_f) \lambda^{af} + (a - a^a \tau^a) \tilde{\psi} \right] &= 0, \\
i \nabla \tilde{\psi} + \frac{i}{\sqrt{2}} \left[ \lambda_f q^f + \lambda_f^a (\tau^a q^f) + (a + a^a \tau^a) \psi \right] &= 0.
\end{aligned} \tag{3.23}$$

Now we substitute the supertranslational and superorientational fermion zero modes (3.18) and (3.20) into these equations in the limit  $\mu = 0$ . After some algebra we managed to check that they do satisfy the Dirac equations (3.22) and (3.23) provided the first-order equations for the string profile functions (3.2) are fulfilled (this check for superorientational modes was done in [11]).

### 3.2 $CP(1) \times C$ model on the string worldsheet: direct calculation in the $\mathcal{N} = 2$ limit

The zero modes (3.18) and (3.20) generate the fermion part of the  $\mathcal{N} = (2, 2)$  model with the target space  $CP(1) \times C$ . This statement was checked in [11]. To perform the check we assume, as usual, that the fermion collective coordinates  $\zeta_{L,R}$  and  $\chi_{L,R}^a$  have

an adiabatic dependence on the worldsheet coordinates  $x_k$  ( $k = 0, 3$ ). Substituting (3.18) and (3.20) in the fermion kinetic terms in the bulk theory (3.14), and taking into account the derivatives of  $\zeta_{L,R}$  and  $\chi_{L,R}^a$  with respect to the worldsheet coordinates we arrive at

$$S_{1+1} = \int dt dz \left\{ 2\pi\xi \left[ \frac{1}{2} (\partial_k x_{0i})^2 + \frac{1}{2} \bar{\zeta}_R i (\partial_0 - i\partial_3) \zeta_R + \frac{1}{2} \bar{\zeta}_L i (\partial_0 + i\partial_3) \zeta_L \right] \right. \\ \left. + \beta \left[ \frac{1}{2} (\partial_k S^a)^2 + \frac{1}{2} \chi_R^a i (\partial_0 - i\partial_3) \chi_R^a + \frac{1}{2} \chi_L^a i (\partial_0 + i\partial_3) \chi_L^a - \frac{1}{2} (\chi_R^a \chi_L^a)^2 \right] \right\}, \quad (3.24)$$

where  $x_{0i}$  ( $i = 1, 2$ ) denote the coordinates of the string center in  $(1, 2)$ -plane; the value of  $\beta$  is determined by the same integral (3.11) as in the bosonic kinetic term, see Eq. (3.10). The first line corresponds to the  $C$  part of the target space, while the second line to the  $CP(1)$  part. The model specified by the second line in Eq. (3.24) (plus the constraint (3.21)) is also known as supersymmetric  $O(3)$  sigma model [24, 25]. In the  $\mathcal{N} = 2$  limit all three fields,  $x_0$ ,  $\zeta_L$  and  $\zeta_R$  are sterile. Deformations to be discussed below will leave  $x_0$  and  $\zeta_L$  sterile, while at the same time will couple  $\zeta_R$  with the  $CP(1)$  sector.

In fact, our derivation gives only the quadratic terms in the fermion fields. The four-fermion term is not accessible in this approximation. The worldsheet  $\mathcal{N} = (2, 2)$  supersymmetry was used in [11] to reconstruct the four-fermion interactions inherent to  $CP(1)$ . The SUSY transformations in the  $CP(1)$  model have the form (see e.g. [10])

$$\begin{aligned} \delta\chi_R^a &= i\sqrt{2} (\partial_0 + i\partial_3) S^a \varepsilon_2 + \sqrt{2}\varepsilon_1 S^a (\chi_R^a \chi_L^a), \\ \delta\chi_L^a &= i\sqrt{2} (\partial_0 - i\partial_3) S^a \varepsilon_1 - \sqrt{2}\varepsilon_2 S^a (\chi_R^a \chi_L^a), \\ \delta S^a &= \sqrt{2}(\varepsilon_1 \chi_L^a + \varepsilon_2 \chi_R^a), \end{aligned} \quad (3.25)$$

where  $\varepsilon_{1,2}$  are two parameters of extended  $\mathcal{N} = (2, 2)$  transformations (for simplicity we restrict ourselves to the real parts of these transformations). Imposing this supersymmetry fixes the four-fermion term in (3.24).

$CP(N-1)$  model (which is a part of string worldsheet theory) can be nicely rewritten in terms of two-dimensional  $U(1)$  gauge theory of  $N$  complex fields  $n^l$  in the strong coupling ( $e^2 \rightarrow \infty$ ) limit [26]. This is the so-called gauged formulation in which the bosonic part of the action takes the form

$$\begin{aligned} S_{CP(1)\text{ bos}} &= \int d^2x \left\{ |\nabla_k n^l|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 + \frac{1}{2e^2} D^2 \right. \\ &\quad \left. + 2|\sigma|^2 |n^l|^2 + iD(|n^l|^2 - 2\beta) \right\}, \end{aligned} \quad (3.26)$$

where

$$\nabla_k = \partial_k - iA_k, \quad F_{kl} = \partial_k A_l - \partial_l A_k,$$

while  $\sigma$  is a complex scalar field and  $D$  is the  $D$ -component of the gauge multiplet. Eliminating the  $D$ -component leads to the constraint

$$|n^l|^2 = 2\beta. \quad (3.27)$$

Moreover, in the limit  $e^2 \rightarrow \infty$  the gauge field  $A_k$  and its  $\mathcal{N} = 2$  bosonic superpartner  $\sigma$  become auxiliary (their kinetic terms vanish) and can be eliminated by virtue of the equations of motion (the fermion fields are ignored so far),

$$A_k = -\frac{i}{4\beta} \bar{n}_l \overleftrightarrow{\partial}_k n^l, \quad \sigma = 0. \quad (3.28)$$

With  $2N$  complex fields  $n^l$ , one real constraint (3.27) and one phase “eaten” by gauging  $U(1)$ , the model has  $2N - 1 - 1 = 2(N - 1)$  real degrees of freedom.

At  $N = 2$  Eq. (3.26) is equivalent to the bosonic action of  $O(3)$  sigma model (3.10). The relation between the variables  $S^a$  and  $n^l$  is

$$S^a = \frac{1}{2\beta} \bar{n} \tau^a n. \quad (3.29)$$

The fermionic part of the  $CP(N - 1)$  model action written in the gauged formulation has the form

$$\begin{aligned} S_{CP(1) \text{ ferm}} &= \int d^2x \left\{ \bar{\xi}_{lR} i(\nabla_0 - i\nabla_3) \xi_R^l + \bar{\xi}_{lL} i(\nabla_0 + i\nabla_3) \xi_L^l \right. \\ &+ \frac{1}{e^2} \bar{\lambda}_R i(\nabla_0 - i\nabla_3) \lambda_R + \frac{1}{e^2} \bar{\lambda}_L i(\nabla_0 + i\nabla_3) \lambda_L + i\sqrt{2} \sigma \bar{\xi}_{lR} \xi_L^l \\ &\left. + i\sqrt{2} \bar{n}_l (\lambda_R \xi_L^l - \lambda_L \xi_R^l) + \text{c.c.} \right\}, \end{aligned} \quad (3.30)$$

where the fields  $\xi_{L,R}^l$  are fermion superpartners of  $n^l$  while  $\lambda_{L,R}$  belong to the gauge multiplet. In the limit  $e^2 \rightarrow \infty$  the fields  $\lambda_{L,R}$  become auxiliary implying the following constraints:

$$\bar{n}^l \xi_L^l = 0, \quad \bar{n}^l \xi_R^l = 0. \quad (3.31)$$

With the fermions switched on Eq. (3.28) must be replaced by

$$\begin{aligned} A_0 + iA_3 &= -\frac{i}{4\beta} \bar{n}_l \left( \overleftrightarrow{\partial}_0 + i \overleftrightarrow{\partial}_3 \right) n^l - \frac{1}{2\beta} \bar{\xi}_R \xi_R, \\ A_0 - iA_3 &= -\frac{i}{4\beta} \bar{n}_l \left( \overleftrightarrow{\partial}_0 - i \overleftrightarrow{\partial}_3 \right) n^l - \frac{1}{2\beta} \bar{\xi}_L \xi_L, \\ \sigma &= -\frac{i}{2\sqrt{2}\beta} \bar{\xi}_{lL} \xi_R^l. \end{aligned} \quad (3.32)$$

The extra  $\xi$  terms in  $A_k$  and  $\sigma$  are responsible for the four-fermion part of the Lagrangian in the gauged formulation.

At  $N = 2$ , the theory specified in (3.26), (3.30) and (3.32) is equivalent to the  $O(3)$  sigma model (3.24). The relation between the complex fermions  $\xi^l$  of the gauged formulation and real fermions  $\chi^a$  of the  $O(3)$  sigma model is

$$\chi_{L,R}^a = \frac{1}{2\beta} \left( \bar{n} \tau^a \xi_{L,R} + \bar{\xi}_{L,R} \tau^a n \right). \quad (3.33)$$

## 4 Digression: Edalati–Tong’s suggestion

The previous part of the paper, along with new results for the fermion zero modes, contained many elements of a review nature. Now we are finally ready to venture into uncharted waters which will bring us, eventually, to “heterotic  $CP(1)$ .”

Let us break  $\mathcal{N} = 2$  supersymmetry of the bulk theory by switching on the deformation superpotential of the type (2.2) or (2.24). In both cases the field  $\tilde{q}$  has vanishing VEV, and the string solutions remains BPS-saturated [11, 13].

The case of the adjoint mass deformation (2.2) was considered in detail in [11]. With four supercharges of the deformed  $\mathcal{N} = 1$  bulk theory normally the 1/2 BPS-saturated string solution (3.7) will preserve only two supercharges on the string worldsheet. However, the number of the fermion zero modes on the string does not change when we break  $\mathcal{N} = 2$  by virtue of the superpotential (2.2). This number is fixed by the index theorem obtained in [27]. Thus, the number of (classically) massless fermion fields on the worldsheet does not change. It is well-known that the sigma model with the  $CP(N-1)$  target space, when supersymmetrized, automatically yields  $\mathcal{N} = (2, 2)$  sigma model; one cannot get  $\mathcal{N} = (0, 2)$ . Therefore, in Ref. [11] it was concluded that the worldsheet theory has an “accidental” SUSY enhancement.

On the other hand, in the recent publication [13] it was pointed out that the target space in the problem at hand is  $CP(N-1) \times C$  rather than  $CP(N-1)$ . Edalati and Tong suggested that the superorientational zero modes can mix with supertranslational ones. In fact, even earlier it was noted [28] that such a mixing, if it takes place, could occur only through a modification of the constraint (3.31),

$$\bar{n}^l \xi_R^l \propto \bar{\zeta}_R$$

in the case of the monomial deformation (2.2). Moreover, Edalati and Tong explicitly constructed an  $\mathcal{N} = (0, 2)$  supergeneralization of the sigma model with the target space  $CP(N-1) \times C$ . In their construction the  $\mathcal{N} = (2, 2)$  model (3.26), (3.30) is

supplemented by the term

$$\delta S_{1+1} = \int d^2x \, 2\beta \left\{ 4 \left| \frac{\partial \mathcal{W}_{1+1}}{\partial \sigma} \right|^2 + i m_W \bar{\lambda}_L \frac{\partial^2 \mathcal{W}_{1+1}}{\partial \sigma^2} \zeta_R \right\} \quad (4.1)$$

breaking  $\mathcal{N} = (2, 2)$  down to  $\mathcal{N} = (0, 2)$ . Here  $\mathcal{W}_{1+1}$  is a two-dimensional deformation superpotential while  $m_W$  is the mass of the  $SU(2)$  gauge boson (2.12). Integrating out the axillary field  $\lambda$  now leads us to

$$\bar{n}^l \xi_L^l = 0, \quad \bar{\xi}_{lR} n^l = \sqrt{2}\beta m_W \frac{\partial^2 \mathcal{W}_{1+1}}{\partial \sigma^2} \zeta_R. \quad (4.2)$$

The left-handed fermion sector remains intact, while the right-handed fermion sector changes. The constraint (3.31) is modified: the right-handed fermion  $\zeta_R$  from the translational sector no longer decouples from the orientational one. The left-handed fermion  $\zeta_L$  as well as  $x_0$  remain free fields and can be omitted in what follows. We suggest a concise name for the model obtained in this way: “heterotic  $CP(1)$ .” The general structure of the deformation (4.1) is dictated by  $\mathcal{N} = (0, 2)$  supersymmetry.

Our analysis fully confirms the above statements. However, this is not the end of the story: Edalati and Tong suggested, additionally, that the bulk and worldsheet deformation superpotentials coincide,

$$\mathcal{W}_{1+1} \sim \mathcal{W}_{3+1}, \quad (4.3)$$

implying that this coincidence takes place for all superpotentials of the type (2.2) and (2.24). The analysis to be presented below shows that Eq. (4.3) is valid only at small  $\mu$ , to the leading order in  $\mu$ . To this order the worldsheet theory deformation is determined essentially by the critical points of the superpotential. At finite or large  $\mu$  the worldsheet deformation superpotential is not given by the simple formula (4.3). In particular, for the deformation (2.24) it becomes nonpolynomial. In Sects. 5–8 we study the bulk deformation (2.2) and then in Sect. 9 turn to the bulk deformation (2.24). In the remainder of this paper we will derive the string worldsheet theory starting from the  $\mathcal{N} = 1$  bulk theory with the deformation superpotentials (2.2) or (2.24).

Let us first dwell on (2.2) which implies  $\mathcal{W}_{1+1} \propto \sigma^2$ . The gauged formulation exploited by Edalati and Tong is convenient for establishing a general structure of the two-dimensional  $\mathcal{N} = (0, 2)$  sigma model. However, it is inconvenient if one’s goal is a direct derivation of this worldsheet model from the bulk theory. The reason is rather obvious: in the gauged formulation both the bosonic part and two-fermion terms in the worldsheet Lagrangian are the same for  $\mathcal{N} = (2, 2)$  and  $\mathcal{N} = (0, 2)$ . Therefore, to detect the difference one has to deal with four-fermion terms whose

extraction from the bulk theory is technically very difficult. At the same time, as we will see shortly, in the  $O(3)$  formulation with the undeformed constraints (3.21) the difference between  $\mathcal{N} = (2, 2)$  and  $\mathcal{N} = (0, 2)$  shows up in two-fermion terms which are readily calculable from the bulk theory given our knowledge of the fermion zero modes.

In the next section we will prove the above statement by deriving  $\mathcal{N} = (0, 2)$  supergeneralization of the  $O(3)$  sigma model. To this end we will need to redefine the field  $\xi_R$  by introducing a linear combination of  $\xi_R$  and  $\zeta_R$  such that for the new field  $\xi'_R = \xi_R - \text{const } n \bar{\zeta}_R$  we have the conventional constraint  $\bar{n} \xi'_R = 0$ . Then we can use Eqs. (3.29) and (3.33) (with  $\xi$  replaced by  $\xi'$ ) to pass to the  $O(3)$  formulation. The constraint  $\vec{S} \vec{\chi}_{L,R} = 0$  will be satisfied automatically. In Sect. 5 we will present the result of this construction assuming that the deformation superpotential is that of the mass term.

## 5 Heterotic $CP(1)$

Here we will derive the  $\mathcal{N} = (0, 2)$  supergeneralization of the  $O(3)$  sigma model following the program outlined above. We will assume for the time being that the deformation potential is monomial, see Eq. (2.2), or equivalently,<sup>4</sup>

$$\mathcal{W}_{1+1} = \frac{\delta}{2} \sigma^2, \quad (5.1)$$

where  $\delta$  is a constant (see below and Appendix D). Performing a rather straightforward algebraic analysis we get

$$\begin{aligned} S_{1+1} = & \int d^2x \left\{ 2\pi\xi \left[ \frac{1}{2} (\partial_k \vec{x}_0)^2 + \frac{1}{2} \bar{\zeta}_L i \partial_R \zeta_L + \frac{1}{2} \bar{\zeta}_R i \partial_L \zeta_R \right] \right. \\ & + \beta \left[ \frac{1}{2} (\partial_k S^a)^2 + \frac{1}{2} \chi_R^a i \partial_L \chi_R^a + \frac{1}{2} \chi_L^a i \partial_R \chi_L^a - \frac{c^2}{2} (\chi_R^a \chi_L^a)^2 \right. \\ & + c \chi_R^a \left( i \partial_L S^a (\kappa \zeta_R + \bar{\kappa} \bar{\zeta}_R) + i \varepsilon^{abc} S^b i \partial_L S^c (\kappa \zeta_R - \bar{\kappa} \bar{\zeta}_R) \right) \\ & \left. \left. + 2 |\kappa|^2 c^2 \bar{\zeta}_R \zeta_R i \varepsilon^{abc} S^a \chi_L^b \chi_L^c \right] \right\}, \quad (5.2) \end{aligned}$$

---

<sup>4</sup>The relation between  $\delta$  and  $\mu$  will be established in Sect. 8. According to the Edalati–Tong conjecture  $\delta \sim \mu$ . We will see that at small  $\mu$  proportionality between  $\mu$  and  $\delta$  does take place. At large  $\mu$ , as we will see below,  $\delta$  grows as  $\ln \mu$ .

where the vector  $\vec{x}$  parametrizes the position of the flux tube center in the perpendicular plane,

$$\partial_L \equiv \partial_0 - i\partial_3, \quad \partial_R \equiv \partial_0 + i\partial_3, \quad (5.3)$$

and

$$\begin{aligned} c^2 &= \frac{1}{1 + |\alpha|^2}, \\ \alpha &\equiv \frac{2\sqrt{2}\kappa}{m_W} = \frac{2\sqrt{2}\kappa}{g_2\sqrt{\xi}}. \end{aligned} \quad (5.4)$$

The relation of the deformation parameter introduced here and the one in the gauged formulation of the theory (see Eqs. (4.1) and (5.1)) is as follows:

$$\delta = \frac{\alpha}{\sqrt{1 - |\alpha|^2}}. \quad (5.5)$$

To simplify reading of the paper we summarize relations between different definitions of the deformation parameters of the worldsheet theory in Appendix D.

As was mentioned, the constraints  $S^a \chi_L^a = 0$ ,  $S^a \chi_R^a = 0$  for fermions  $\chi$  stay intact. This is achieved through shifting the field  $\xi_R$ ,

$$\bar{\xi}_R \rightarrow \bar{\xi}_R - \delta \frac{m_W}{\sqrt{2}} \bar{n} \zeta_R, \quad (5.6)$$

see Eq. (4.2). As a result, crucial bifermionic terms of the type  $\chi_R \partial S \zeta_R$  appear in the third line of Eq. (5.2).

Generically the parameter  $\kappa$  is complex. Its phase is determined by the bulk deformation parameter  $\mu$ ,

$$\arg \kappa = \arg \mu. \quad (5.7)$$

Later on, for simplicity, we assume that the bulk parameter  $\mu$  is real; therefore,  $\kappa$  is real too.

The two-dimensional fields  $\vec{x}_0$  and  $\zeta_L$  forming a representation of  $\mathcal{N} = (0, 2)$  superalgebra are sterile; they are decoupled from all other fields in the action (5.2). Although these fields are a part of the string worldsheet theory they play no dynamical role. Therefore, in discussing dynamical aspects of the heterotic model under consideration we can (and will) safely omit them in what follows. Then the two-dimensional Lagrangian takes the form

$$L_{1+1} = \beta \left[ \frac{1}{2} (\partial_k S^a)^2 + \frac{1}{2} \chi_R^a i \partial_L \chi_R^a + \frac{1}{2} \chi_L^a i \partial_R \chi_L^a - \frac{c^2}{2} (\chi_R^a \chi_L^a)^2 \right]$$



$$\begin{aligned}
& + \pi \xi \left( \bar{\zeta}_R i \partial_L \zeta_R \right) + \beta \left[ c \kappa \chi_R^a \left( i \partial_L S^a (\zeta_R + \bar{\zeta}_R) + i \varepsilon^{abc} S^b i \partial_L S^c (\zeta_R - \bar{\zeta}_R) \right) \right. \\
& \left. + 2 \kappa^2 c^2 \bar{\zeta}_R \zeta_R i \varepsilon^{abc} S^a \chi_L^b \chi_L^c \right].
\end{aligned} \tag{5.8}$$

The first line represents the conventional  $\mathcal{N} = (2, 2)$   $O(3)$  sigma model (if we put  $c = 1$ ), while the second and the third lines give its  $\mathcal{N} = (0, 2)$  deformation. The  $\pi \xi$  normalizing factor in front of the kinetic term  $\bar{\zeta}_R i \partial_L \zeta_R$  is due to “historical” reasons. The field  $\zeta_R$  used to be a superpartner of  $\vec{x}_0$  in  $\mathcal{N} = (2, 2)$ . As well-known, the normalization of the kinetic term of  $\vec{x}_0$  is given by the string tension. In Sect. 6 we will switch to the canonic normalization of the kinetic term  $\bar{\zeta}_R i \partial_L \zeta_R$ . The constant  $\beta$  is related to the bulk constant  $g_2^2$  and the conventionally normalized  $O(3)$  model constant  $g_0^2$  (see Sect. 6) as follows:

$$\beta = \frac{2\pi}{g_2^2} = \frac{1}{g_0^2}. \tag{5.9}$$

Let us ask ourselves: how many independent constants characterize the heterotic  $O(3)$  sigma model besides  $g_0^2$ ? At first sight, Eq. (5.8) contains two constants,  $\xi$  and  $\kappa$ . In fact, there is only one extra constant. This is readily seen if one rescales the fields  $\zeta_R$  to make their kinetic term canonically normalized. Then one immediately sees that, besides  $\beta = g_0^{-2}$ , Eq. (5.8) contains a single additional constant,  $\alpha$ . This fact will be demonstrated again in Sect. 6 where an alternative derivation of the heterotic deformation of the  $CP(1)$  model is given. The relation between the (only) deformation parameter  $\gamma$  introduced in Sect. 6 and  $\alpha$  is as follows:

$$\gamma^2 = \frac{1}{g_0^2} \frac{\alpha^2}{1 + \alpha^2} = \beta \frac{\alpha^2}{1 + \alpha^2}. \tag{5.10}$$

We assume  $\gamma$  to be real.

Concluding this section we present modified supertransformations. The model (5.2) is invariant under the following  $\mathcal{N} = (0, 2)$  SUSY transformations:

$$\begin{aligned}
\delta \chi_R^a &= \sqrt{2} \varepsilon_1 S^a (\chi_R^b \chi_L^b) - \sqrt{2} \kappa c \varepsilon_1 \left[ \chi_L^a (\zeta_R + \bar{\zeta}_R) \right. \\
&\quad \left. + i \varepsilon^{abc} S^b \chi_L^c (\zeta_R - \bar{\zeta}_R) \right], \\
\delta \chi_L^a &= i \sqrt{2} (\partial_0 - i \partial_3) S^a \varepsilon_1, \\
\delta S^a &= \sqrt{2} \varepsilon_1 \chi_L^a, \\
\delta \zeta_L &= i \sqrt{2} [(\partial_0 - i \partial_3) z] \varepsilon_1, \quad z = x_{01} + i x_{02},
\end{aligned}$$

$$\begin{aligned}\delta z &= \sqrt{2}\varepsilon_1\zeta_L, \\ \delta\zeta_R &= -2c\sqrt{2}\frac{\kappa}{m_W^2}\varepsilon_1\left[\chi_R^a\chi_L^a + i\varepsilon^{abc}S^a\chi_R^b\chi_L^c\right],\end{aligned}\tag{5.11}$$

where  $z$  is the complexified coordinate of the string center. Here, much in the same way as in (3.25), we present for simplicity the SUSY transformations generated only by the real parameter  $\varepsilon_1$ . Note that we no longer have  $\varepsilon_2$ -transformations in our worldsheet theory. These are broken by the  $\mathcal{N} = (0, 2)$  deformation.

## 6 Geometric formulation of the $\mathcal{N} = (0, 2)$ heterotic model

In this section we will derive the deformed  $CP(1)$  model with no reference to string worldsheet theory and arguments in [13], through an appropriately modified superfield formalism. At the end we will present the Lagrangian of the heterotic  $CP(N-1)$  model with arbitrary  $N$ .

**Warning:** In this section, unlike others, we use the Minkowski notation and normalize the kinetic term of  $\zeta_R$  canonically.

To begin with, let us outline the standard geometric formulation of the supersymmetric  $CP(N-1)$  sigma model in 1+1 dimensions,  $x^\mu = \{t, z\}$  [9, 10], with the subsequent deformation down to  $\mathcal{N} = (0, 2)$ .

The target space is the  $(N-1)$ -dimensional Kähler manifold parametrized by the fields  $\phi^i, \phi^{\dagger\bar{j}}, i, \bar{j} = 1, \dots, N-1$ , which are the lowest components of the chiral and antichiral superfields

$$\begin{aligned}\Phi^i(x^\mu + i\bar{\theta}\gamma^\mu\theta), \quad \Phi^{\dagger\bar{j}}(x^\mu - i\bar{\theta}\gamma^\mu\theta), \\ \Phi^i = \phi^i + \sqrt{2}\theta\psi^i + \theta^2 F^i.\end{aligned}\tag{6.12}$$

The Lagrangian is [29]

$$\mathcal{L} = \int d^4\theta K(\Phi, \Phi^\dagger) = G_{i\bar{j}}[\partial_\mu\phi^{\dagger\bar{j}}\partial_\mu\phi^i + i\bar{\psi}^{\bar{j}}\gamma^\mu D_\mu\psi^i] - \frac{1}{2}R_{i\bar{j}k\bar{l}}(\bar{\psi}^{\bar{j}}\psi^i)(\bar{\psi}^{\bar{l}}\psi^k),\tag{6.13}$$

where  $K(\Phi, \Phi^\dagger)$  is the Kähler potential,

$$G_{i\bar{j}} = \frac{\partial^2 K(\phi, \phi^\dagger)}{\partial\phi^i\partial\phi^{\dagger\bar{j}}}$$

is the Kähler metric,  $R_{i\bar{j}k\bar{l}}$  is the Riemann tensor,

$$D_\mu\psi^i = \partial_\mu\psi^i + \Gamma_{kl}^i\partial_\mu\phi^k\psi^l$$

is the covariant derivative acting on the fermion field. Our choice of the gamma-matrices is summarized in Appendix B, along with some other definitions.

A particular choice of the Kähler potential

$$K = \frac{2}{g_0^2} \ln \left( 1 + \sum_{i,\bar{j}=1}^{N-1} \Phi^{\dagger\bar{j}} \delta_{\bar{j}i} \Phi^i \right) \quad (6.14)$$

is most common, it corresponds to the round Fubini–Study metric. For  $CP(N-1)$ , the Ricci tensor  $R_{i\bar{j}}$  is proportional to the metric,

$$R_{i\bar{j}} = \frac{g_0^2}{2} N G_{i\bar{j}}, \quad (6.15)$$

see also (6.40).

The conserved supercurrent is

$$J_\alpha^\mu = \sqrt{2} G_{i\bar{j}} [\partial_\nu \phi^{\dagger\bar{j}} \gamma^\nu \gamma^\mu \psi^i]_\alpha. \quad (6.16)$$

In terms of the  $R, L$  components it takes the form

$$\begin{aligned} J_R^0 &= J_R^1 = \sqrt{2} G_{i\bar{j}} [\partial_R \phi^{\dagger\bar{j}}] \psi_R^i, \\ J_L^0 &= -J_L^1 = \sqrt{2} G_{i\bar{j}} [\partial_L \phi^{\dagger\bar{j}}] \psi_L^i, \end{aligned} \quad (6.17)$$

where

$$\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \quad \partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}, \quad (6.18)$$

and

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}. \quad (6.19)$$

Then the superconformal anomaly can be expressed as follows:

$$\begin{aligned} J_{\text{sc}} &\equiv \gamma_\mu J_\alpha^\mu = \frac{\sqrt{2}}{2\pi} R_{i\bar{j}} [\partial_\nu \phi^{\dagger\bar{j}}] \gamma^\nu \psi^i, \\ J_{\text{sc},R} &= \frac{-i\sqrt{2}}{2\pi} R_{i\bar{j}} [\partial_R \phi^{\dagger\bar{j}}] \psi_L^i, \quad J_{\text{sc},L} = \frac{i\sqrt{2}}{2\pi} R_{i\bar{j}} [\partial_L \phi^{\dagger\bar{j}}] \psi_R^i. \end{aligned} \quad (6.20)$$

For what follows it is helpful to collect here explicit expressions in the case of  $CP(1)$ . In this case a single complex field  $\phi(t, z)$  serves as coordinate on the target space which is equivalent to  $S^2$ . The Kähler potential  $K$ , the metric  $G$ , the Christoffel symbols  $\Gamma, \bar{\Gamma}$  and the Ricci tensor  $R$  are

$$K|_{\theta=\bar{\theta}=0} = \frac{2}{g_0^2} \ln \chi,$$

$$\begin{aligned}
G &= G_{1\bar{1}} = \partial_\phi \partial_{\phi^\dagger} K|_{\theta=\bar{\theta}=0} = \frac{2}{g_0^2 \chi^2}, \\
\Gamma &= \Gamma_{11}^1 = -2 \frac{\phi^\dagger}{\chi}, \quad \bar{\Gamma} = \Gamma_{\bar{1}\bar{1}}^{\bar{1}} = -2 \frac{\phi}{\chi}, \\
R &\equiv R_{1\bar{1}} = -G^{-1} R_{1\bar{1}1\bar{1}} = \frac{2}{\chi^2},
\end{aligned} \tag{6.21}$$

where we use the notation

$$\chi \equiv 1 + \phi \phi^\dagger. \tag{6.22}$$

The Lagrangian of the conventional  $CP(1)$  model takes the form

$$L_{CP(1)} = G \left\{ \partial_\mu \phi^\dagger \partial^\mu \phi + i \bar{\psi} \gamma^\mu \partial_\mu \psi - \frac{2i}{\chi} \phi^\dagger \partial_\mu \phi \bar{\psi} \gamma^\mu \psi + \frac{1}{\chi^2} (\bar{\psi} \psi)^2 \right\}. \tag{6.23}$$

In terms of the components  $\psi_{L,R}$ , the Lagrangian (6.23) can be rewritten as

$$\begin{aligned}
L_{CP(1)} &= G \left\{ \partial_\mu \phi^\dagger \partial^\mu \phi + \frac{i}{2} (\psi_L^\dagger \overleftrightarrow{\partial}_R \psi_L + \psi_R^\dagger \overleftrightarrow{\partial}_L \psi_R) \right. \\
&\quad \left. - \frac{i}{\chi} [\psi_L^\dagger \psi_L (\phi^\dagger \overleftrightarrow{\partial}_R \phi) + \psi_R^\dagger \psi_R (\phi^\dagger \overleftrightarrow{\partial}_L \phi)] - \frac{2}{\chi^2} \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \right\},
\end{aligned} \tag{6.24}$$

where  $\partial_{L,R} = \partial/\partial t \pm \partial/\partial z$ , see Appendix B. It is not difficult to check that it has  $\mathcal{N} = (2, 2)$  supersymmetry.

Now, let us introduce a heterotic deformation, due to the right-handed field  $\zeta_R$ , which transforms the above  $\mathcal{N} = (2, 2)$  model into  $\mathcal{N} = (0, 2)$ ,

$$\begin{aligned}
L_{\text{heterotic}} &= \zeta_R^\dagger i \partial_L \zeta_R + \left[ \gamma \zeta_R R (i \partial_L \phi^\dagger) \psi_R + \text{H.c.} \right] - g_0^2 |\gamma|^2 (\zeta_R^\dagger \zeta_R) (R \psi_L^\dagger \psi_L) \\
&+ G \left\{ \partial_\mu \phi^\dagger \partial^\mu \phi + \frac{i}{2} (\psi_L^\dagger \overleftrightarrow{\partial}_R \psi_L + \psi_R^\dagger \overleftrightarrow{\partial}_L \psi_R) \right. \\
&\quad \left. - \frac{i}{\chi} [\psi_L^\dagger \psi_L (\phi^\dagger \overleftrightarrow{\partial}_R \phi) + \psi_R^\dagger \psi_R (\phi^\dagger \overleftrightarrow{\partial}_L \phi)] - \frac{2(1 - g_0^2 |\gamma|^2)}{\chi^2} \psi_L^\dagger \psi_L \psi_R^\dagger \psi_R \right\},
\end{aligned} \tag{6.25}$$

where  $R$  stands for the Ricci tensor, see Eq. (6.21). One can obtain the deformed Lagrangian (6.25) as follows. Introduce the operators

$$\begin{aligned}
\mathcal{B} &= \left\{ \zeta_R (x^\mu + i \bar{\theta} \gamma^\mu \theta) + \sqrt{2} \theta_R \mathcal{F} \right\} \theta_L^\dagger, \\
\mathcal{B}^\dagger &= \theta_L \left\{ \zeta_R^\dagger (x^\mu - i \bar{\theta} \gamma^\mu \theta) + \sqrt{2} \theta_R^\dagger \mathcal{F}^\dagger \right\}.
\end{aligned} \tag{6.26}$$

Since  $\theta_L$  and  $\theta_L^\dagger$  enter in Eq. (6.26) explicitly,  $\mathcal{B}$  and  $\mathcal{B}^\dagger$  are *not* superfields with regards to the supertransformations with parameters  $\epsilon_L, \epsilon_L^\dagger$ . These supertransformations are

absent in the heterotic model. Only those survive which are associated with  $\epsilon_R, \epsilon_R^\dagger$ . Note that  $\mathcal{B}$  and  $\mathcal{B}^\dagger$  are superfields with regards to the latter. Here  $\epsilon_R$  is related to the real parameter of supersymmetry transformations we dealt with in previous sections as  $\text{Re } \epsilon_R = \varepsilon_1$ .

It is convenient to introduce a shorthand for the chiral coordinate

$$\tilde{x}^\mu = x^\mu + i\bar{\theta}\gamma^\mu\theta. \quad (6.27)$$

Then the transformation laws with the parameters  $\epsilon_R, \epsilon_R^\dagger$  are as follows:

$$\delta\theta_R = \epsilon_R, \quad \delta\theta_R^\dagger = \epsilon_R^\dagger, \quad \delta\tilde{x}^0 = 2i\epsilon_R^\dagger\theta_R, \quad \delta\tilde{x}^1 = 2i\epsilon_R^\dagger\theta_R. \quad (6.28)$$

With respect to such supertransformations,  $\mathcal{B}$  and  $\mathcal{B}^\dagger$  are superfields. Indeed,

$$\delta\zeta_R = \sqrt{2}\mathcal{F}\epsilon_R, \quad \delta\mathcal{F} = \sqrt{2}i(\partial_L\zeta_R)\epsilon_R^\dagger, \quad (6.29)$$

plus Hermitian conjugate transformations. To convert  $L_{CP(1)}$  into  $L_{\text{heterotic}}$  we add to  $L_{CP(1)}$  the following terms:

$$\Delta L = \int d^4\theta \left\{ -2\mathcal{B}^\dagger\mathcal{B} + \left[ g_0^2\sqrt{2}\gamma\mathcal{B}K + \text{H.c.} \right] \right\}, \quad (6.30)$$

where  $\gamma$  is generally speaking a complex constant. For simplicity we will assume  $\gamma$  to be real. Thus, we obviously deal here with a single deformation parameter. Its relation to  $\alpha$  is discussed in Sect. 5 (see Eq. (5.10)) while the relation to the bulk theory parameters in Sect. 8. To derive Eq. (5.10) it is sufficient to compare the  $\zeta_R\partial_L\phi^\dagger\psi_R$  term in Eq. (6.25) with the  $\chi_R^a\partial_LS^a(\zeta_R + \bar{\zeta}_R)$  term in Eq. (5.8).

First, let us check that the extra term (6.30) preserves invariance on the target space. Indeed, the invariance under the  $U(1)$  transformation of the superfields  $\Phi, \Phi^\dagger$ ,

$$\Phi \rightarrow i\delta\Phi, \quad \Phi^\dagger \rightarrow -i\delta\Phi^\dagger. \quad (6.31)$$

is obvious. Two other rotations on the sphere manifest themselves in nonlinear transformations with a complex parameter  $\beta$ ,

$$\Phi \rightarrow \beta + \beta^*\Phi^2, \quad \Phi^\dagger \rightarrow \beta^* + \beta(\Phi^\dagger)^2. \quad (6.32)$$

Under these transformations

$$\delta K = \frac{2}{g_0^2} (\beta^*\Phi + \beta\Phi^\dagger). \quad (6.33)$$

It is not difficult to see that

$$\int d^4\theta \mathcal{B}\delta K = 0. \quad (6.34)$$

In other words, even before performing the component decomposition we are certain that the term (6.30) is invariant on the target space of the  $CP(1)$  model. Needless to say, it is  $\mathcal{N} = (0, 2)$  invariant by construction.

As usual, the  $\mathcal{F}$  term enters without derivatives and can be eliminated by virtue of equations of motion,

$$\mathcal{F} = -2\gamma^* \chi^{-2} \psi_R^\dagger \psi_L, \quad \mathcal{F}^\dagger = -2\gamma \chi^{-2} \psi_L^\dagger \psi_R. \quad (6.35)$$

This is responsible for the change of the standard coefficient in front of the four-fermion term (the last line in Eq. (6.25)). As for the target space structure of this coefficient, it is proportional to the curvature tensor of  $CP(1)$ .

In addition, the  $F$  terms of the superfields  $\Phi$ ,  $\Phi^\dagger$  also change. If before the deformation e.g.  $F = (i/2) \Gamma \psi \gamma^0 \psi$ , after the deformation

$$F = \frac{i}{2} \Gamma \psi \gamma^0 \psi - g_0^2 \gamma \psi_L \zeta_R^\dagger, \quad (6.36)$$

plus the Hermitian conjugated expression for  $F^\dagger$ . As a result, we get the last term in the first line in Eq. (6.25).

If we take into account the relation between  $\gamma$  and the bulk theory parameters we can conclude that  $g_0^2 |\gamma|^2 < 1$ .

The last issue to be discussed in this section is the change of the supercurrent. The supercurrent in the conventional  $CP(1)$  model is given in Eq. (6.20). When we deform the model the supercurrent acquires extra terms associated with the  $\mathcal{F}$ ,  $F$  terms in Eqs. (6.35) and (6.36). This term is proportional to  $\gamma \{R \psi_R^\dagger \psi_L\} \zeta_R^\dagger$ , and its Hermitian conjugate, of course. Assume that  $\gamma \ll 1$ . Then the expression in the braces can be evaluated in the undeformed  $CP(1)$  model. As well known (see e.g. [10]), a nonvanishing bifermion condensate  $\langle R \psi_R^\dagger \psi_L \rangle \sim \pm \Lambda$  develops in this model ( $\Lambda$  is the scale parameter) labeling two distinct vacua. Thus, the additional terms in the supercurrent emerging in the deformed theory (at small  $\gamma$ ) have the form

$$\Delta J_{\text{sc}} = \gamma \langle R \psi_R^\dagger \psi_L \rangle \zeta_R^\dagger, \quad \Delta J_{\text{sc}}^\dagger = \gamma \langle R \psi_L^\dagger \psi_R \rangle \zeta_R. \quad (6.37)$$

Since  $\zeta_R$  is strictly massless Eq. (6.37) clearly demonstrates that  $\zeta_R$  is a Goldstino, with the residue  $\langle R \psi_R^\dagger \psi_L \rangle$ . Supersymmetry is spontaneously broken, with the vacuum energy

$$\mathcal{E}_{\text{vac}} = |\gamma|^2 \left| \langle R \psi_R^\dagger \psi_L \rangle \right|^2 \quad (6.38)$$

times a numerical factor, one and the same for both vacua. In the accompanying paper [17] we will obtain a nonvanishing  $\mathcal{E}_{\text{vac}}$  for arbitrary values of  $\gamma$  in heterotically deformed  $CP(N-1)$  models using large  $N$  expansion. The very possibility

of the spontaneous supersymmetry breaking is due to the fact that Witten's index  $I_W$  of the deformed theory vanishes, in sharp contradistinction with the undeformed conventional  $\mathcal{N} = (2, 2)$  model where  $I_W = N$  [16]. Details of this statement are discussed in Appendix C.

Given the geometric representation of the deformed  $CP(1)$  model (6.25) one can suggest a generalization for arbitrary  $N$  (i.e. the  $\mathcal{N} = (0, 2)$  deformed  $CP(N - 1)$  model),

$$\begin{aligned}
L_{\text{heterotic}} &= \zeta_R^\dagger i \partial_L \zeta_R + \left[ \gamma g_0^2 \zeta_R G_{i\bar{j}} (i \partial_L \phi^{\dagger\bar{j}}) \psi_R^i + \text{H.c.} \right] \\
&- g_0^4 |\gamma|^2 \left( \zeta_R^\dagger \zeta_R \right) \left( G_{i\bar{j}} \psi_L^{\dagger\bar{j}} \psi_L^i \right) \\
&+ G_{i\bar{j}} [\partial_\mu \phi^{\dagger\bar{j}} \partial_\mu \phi^i + i \bar{\psi}^{\bar{j}} \gamma^\mu D_\mu \psi^i] \\
&- \frac{g_0^2}{2} \left( G_{i\bar{j}} \psi_R^{\dagger\bar{j}} \psi_R^i \right) \left( G_{k\bar{m}} \psi_L^{\dagger\bar{m}} \psi_L^k \right) \\
&+ \frac{g_0^2}{2} \left( 1 - 2g_0^2 |\gamma|^2 \right) \left( G_{i\bar{j}} \psi_R^{\dagger\bar{j}} \psi_L^i \right) \left( G_{k\bar{m}} \psi_L^{\dagger\bar{m}} \psi_R^k \right), \tag{6.39}
\end{aligned}$$

where we used the fact that for the above Kähler metric

$$R_{i\bar{j}k\bar{m}} = -\frac{g_0^2}{2} \left( G_{i\bar{j}} G_{k\bar{m}} + G_{i\bar{m}} G_{k\bar{j}} \right). \tag{6.40}$$

We assume that  $g_0^{-2}$  is proportional to  $N$  while  $|\gamma|^2 g_0^2$  has no  $N$  dependence. Note that the term in the fourth line is absent in [13].

## 7 From the bulk $\mathcal{N} = 1$ theory to the heterotic deformation of the $CP(1)$ model on the world-sheet

In this section we will obtain fermion zero modes on the non-Abelian string in the bulk theory with the deformation term (2.2). This will allow us, eventually, to calculate the bifermion cross-term  $\bar{\chi} \partial_L S \zeta_R$  directly from the bulk theory and thus determine the parameter  $\kappa$  in terms of the bulk theory parameters. This will completely fix the heterotic model since the overall structure of the Lagrangian is dictated by  $\mathcal{N} = (0, 2)$  supersymmetry.

## 7.1 Fermion zero modes in $\mathcal{N} = 1$ theory

We start from supertranslational fermion zero modes on the string. Superorientational zero modes for the bulk deformation (2.2) were determined in [11] and we just quote the results. With  $\mathcal{N} = 2$  supersymmetry broken, we have only two complex SUSY transformations left. They are generated by parameters  $\epsilon^{11}$  and  $\epsilon^{21}$ , see Sect. 3.1. The latter transformation acts on the string solution trivially while the former gives two unmodified fermion supertranslational zero modes in (3.18) proportional to  $\zeta_L$ . At the same time, the fields  $\lambda^{22}$  and  $\tilde{\psi}_1$  proportional to  $\zeta_R$  are modified. To find them we explicitly solve below the Dirac equations (3.22) and (3.23).

It is easy to check that the modified fermion fields  $\lambda^{22}$  and  $\tilde{\psi}_1$  can be written in the following form:

$$\begin{aligned}\lambda^{22} &= \lambda_{s0} \zeta_R + \lambda_{s1} \frac{x_1 + ix_2}{r} \bar{\zeta}_R, \\ \lambda^{a22} &= \lambda_{t0} S^a \zeta_R + \lambda_{t1} S^a \frac{x_1 + ix_2}{r} \bar{\zeta}_R, \\ \tilde{\psi}_1 &= \frac{1}{2} \frac{x_1 - ix_2}{r} (\psi_{s0} + \tau^a S^a \psi_{t0}) \zeta_R + \frac{1}{2} (\psi_{s1} + \tau^a S^a \psi_{t1}) \bar{\zeta}_R,\end{aligned}\tag{7.1}$$

where we introduced four profile functions  $\lambda(r)$ , and four functions  $\psi(r)$  parameterizing the fermion fields  $\lambda^{22}$  and  $\tilde{\psi}_1$ , respectively. The subscripts  $s$  and  $t$  label the singlet and triplet profile functions with respect to the unbroken global  $SU(2)_{C+F}$ .

Substituting (7.1) in the Dirac equations (3.22) and (3.23) we get equations for the fermion profile functions for  $\lambda$  fermions

$$\begin{aligned}- \frac{d}{dr} \lambda_{s0} + i \frac{g_1^2}{2\sqrt{2}} [(\phi_1 + \phi_2) \psi_{s0} + (\phi_1 - \phi_2) \psi_{t0}] - g_1^2 \mu \lambda_{s1} &= 0, \\ - \frac{d}{dr} \lambda_{s1} - \frac{1}{r} \lambda_{s1} + i \frac{g_1^2}{2\sqrt{2}} [(\phi_1 + \phi_2) \psi_{s1} + (\phi_1 - \phi_2) \psi_{t1}] - g_1^2 \mu \lambda_{s0} &= 0, \\ - \frac{d}{dr} \lambda_{t0} + i \frac{g_2^2}{2\sqrt{2}} [(\phi_1 - \phi_2) \psi_{s0} + (\phi_1 + \phi_2) \psi_{t0}] - g_2^2 \mu \lambda_{t1} &= 0, \\ - \frac{d}{dr} \lambda_{t1} - \frac{1}{r} \lambda_{t1} + i \frac{g_2^2}{2\sqrt{2}} [(\phi_1 - \phi_2) \psi_{s1} + (\phi_1 + \phi_2) \psi_{t1}] - g_2^2 \mu \lambda_{t0} &= 0,\end{aligned}\tag{7.2}$$



and for  $\psi$  fermions

$$\begin{aligned}
\frac{d}{dr}\psi_{s0} + \frac{1}{r}\psi_{s0} - \frac{f}{2r}\psi_{s0} - \frac{f_3}{2r}\psi_{t0} + \frac{i}{\sqrt{2}} [(\phi_1 + \phi_2)\lambda_{s0} + (\phi_1 - \phi_2)\lambda_{t0}] &= 0, \\
\frac{d}{dr}\psi_{t0} + \frac{1}{r}\psi_{t0} - \frac{f}{2r}\psi_{t0} - \frac{f_3}{2r}\psi_{s0} + \frac{i}{\sqrt{2}} [(\phi_1 - \phi_2)\lambda_{s0} + (\phi_1 + \phi_2)\lambda_{t0}] &= 0, \\
\frac{d}{dr}\psi_{s1} - \frac{f}{2r}\psi_{s1} - \frac{f_3}{2r}\psi_{t1} + \frac{i}{\sqrt{2}} [(\phi_1 + \phi_2)\lambda_{s1} + (\phi_1 - \phi_2)\lambda_{t1}] &= 0, \\
\frac{d}{dr}\psi_{t1} - \frac{f}{2r}\psi_{t1} - \frac{f_3}{2r}\psi_{s1} + \frac{i}{\sqrt{2}} [(\phi_1 - \phi_2)\lambda_{s1} + (\phi_1 + \phi_2)\lambda_{t1}] &= 0.
\end{aligned} \tag{7.3}$$

We will solve these equations below in two limits, for small and large values of the deformation parameter  $\mu$ .

As was already mentioned, the superorientational zero modes in the theory with deformation (2.2) were found in [11]. They have the form

$$\begin{aligned}
\lambda^{a22} &= \frac{x_1 + ix_2}{r} \lambda_+(r) [\chi_R^a - i\varepsilon^{abc} S^b \chi_R^c] + \lambda_-(r) [\chi_R^a + i\varepsilon^{abc} S^b \chi_R^c], \\
\bar{\psi}_i^{kA} &= \psi_+(r) \left(\frac{\tau^a}{2}\right)^{kA} [\chi_R^a - i\varepsilon^{abc} S^b \chi_R^c] \\
&+ \frac{x_1 - ix_2}{r} \psi_-(r) \left(\frac{\tau^a}{2}\right)^{kA} [\chi_R^a + i\varepsilon^{abc} S^b \chi_R^c],
\end{aligned} \tag{7.4}$$

where we introduced four profile functions  $\lambda_\pm$  and  $\psi_\pm$  parameterizing the fermion fields  $\lambda^{22}$  and  $\bar{\psi}_i$ . The functions  $\lambda_+$  and  $\psi_+$  are expandable in even powers of  $\mu$  while the functions  $\lambda_-$  and  $\psi_-$  in odd powers of  $\mu$ .

Substituting (7.4) into the Dirac equations (3.22), (3.23) we get following equations for the fermion profile functions:

$$\begin{aligned}
\frac{d}{dr}\psi_+ - \frac{1}{2r}(f - f_3)\psi_+ + i\sqrt{2}\phi_1\lambda_+ &= 0, \\
-\frac{d}{dr}\lambda_+ - \frac{1}{r}\lambda_+ + \frac{f_3}{r}\lambda_+ + i\frac{g_2^2}{\sqrt{2}}\phi_1\psi_+ + g_2^2\mu\lambda_- &= 0, \\
\frac{d}{dr}\psi_- + \frac{1}{r}\psi_- - \frac{1}{2r}(f + f_3)\psi_- + i\sqrt{2}\phi_2\lambda_- &= 0, \\
-\frac{d}{dr}\lambda_- - \frac{f_3}{r}\lambda_- + i\frac{g_2^2}{\sqrt{2}}\phi_2\psi_- + g_2^2\mu\lambda_+ &= 0.
\end{aligned} \tag{7.5}$$

## 7.2 Small- $\mu$ limit

In terms of the profile functions introduced in (7.1) the undeformed translational fermion zero modes (3.18) of the  $\mathcal{N} = (2, 2)$  model are

$$\begin{aligned}\lambda_{s0} &= ig_1^2(\phi_1^2 + \phi_2^2 - 2\xi) + O(\mu^2), \quad \lambda_{t0} = ig_2^2(\phi_1^2 - \phi_2^2) + O(\mu^2), \\ \psi_{s0} &= \frac{\sqrt{2}}{r}[(f + f_3)\phi_1 + (f - f_3)\phi_2] + O(\mu^2), \\ \psi_{t0} &= \frac{\sqrt{2}}{r}[(f + f_3)\phi_1 - (f - f_3)\phi_2] + O(\mu^2).\end{aligned}\tag{7.6}$$

The profile functions  $\lambda_0$  and  $\psi_0$  can be expressed as series in even powers of  $\mu$ .

On the other hand, the functions  $\lambda_1$  and  $\psi_1$  are expandable in odd powers of  $\mu$ . The Dirac equations (7.2) and (7.3) can be easily solved for these functions, to the leading order in  $\mu$ . We consider  $\mu$ -dependent terms in the second and the last equations in (7.2) as perturbations and substitute there the solutions (7.6). Then we have

$$\begin{aligned}\lambda_{s1} &= -\frac{g^2\mu}{2}r\lambda_{s0} + O(\mu^3), \quad \lambda_{t1} = -\frac{g^2\mu}{2}r\lambda_{t0} + O(\mu^3), \\ \psi_{s1} &= -\frac{g^2\mu}{2}r\psi_{s0} + O(\mu^3), \quad \psi_{t1} = -\frac{g^2\mu}{2}r\psi_{t0} + O(\mu^3),\end{aligned}\tag{7.7}$$

where we put  $g_1 = g_2 \equiv g$  for simplicity.

The behavior of the superorientational fermion zero modes in the small- $\mu$  limit was obtained in [11]. The leading contributions to the  $\mu$ -even profile functions are

$$\lambda_+ = \frac{i}{\sqrt{2}}\frac{f_3}{r}\frac{\phi_1}{\phi_2} + O(\mu^2), \quad \psi_+ = \frac{1}{2\phi_2}(\phi_1^2 - \phi_2^2) + O(\mu^2),\tag{7.8}$$

see Eq. (3.20).

Substituting Eq. (7.8) into the last equation in (7.5) we can solve for the leading contributions to the  $\mu$ -odd profile functions. In this way we get [11]

$$\begin{aligned}\lambda_- &= g_2^2\mu\frac{i}{2\sqrt{2}}\left[(f_3 - 1)\frac{\phi_2}{\phi_1} + \frac{\phi_1}{\phi_2}\right] + O(\mu^3), \\ \psi_- &= g_2^2\mu\frac{r}{4\phi_1}(\phi_1^2 - \phi_2^2) + O(\mu^3).\end{aligned}\tag{7.9}$$

Using the boundary conditions (3.3) and (3.4) for the string profile functions it is easy to check that these solutions vanish at  $r \rightarrow \infty$  and are nonsingular at  $r = 0$ .

### 7.3 Large- $\mu$ limit

Now let us dwell on the limit of large  $\mu$ . As was explained in Sect. 2, the fields  $a$ ,  $a^a$  (as well as their fermion counterparts  $\lambda^{\alpha 2}$ ,  $\lambda^{a\alpha 2}$ ) become heavy and can be integrated out. The low-energy theory contains massive U(1) and SU(2) gauge multiplets, with mass (2.11) and (2.12), and four chiral light multiplets, with mass

$$m_L \equiv m_{\text{SU}(2)}^- = m_{\text{U}(1)}^- = \frac{\xi}{\mu}, \quad (7.10)$$

see Eq. (2.20).

Integrating out heavy fields can be performed directly in the superpotential, as in [30, 31, 15], or in the component Lagrangian. To this end one drops the kinetic terms for all heavy fields and solves algebraic equations for these fields. We do it in the fermion sector of the theory in the Dirac equations (3.22) for  $\lambda^{\alpha 2}$  and  $\lambda^{a\alpha 2}$ . First we consider supertranslational zero modes. The large- $\mu$  limit for superorientational modes was considered in [11] and we just quote the corresponding results.

More exactly, we get expressions for the  $\lambda$ -profile functions in terms of the  $\psi$ -profile functions from equations in (7.3). Namely,

$$\begin{aligned} \lambda_{s0} + \lambda_{t0} &= \frac{i\sqrt{2}}{\phi_1} \left[ \frac{d}{dr} \psi_{+0} + \frac{1}{r} \psi_{+0} - \frac{1}{2r} (f + f_3) \psi_{+0} \right], \\ \lambda_{s0} - \lambda_{t0} &= \frac{i\sqrt{2}}{\phi_2} \left[ \frac{d}{dr} \psi_{-0} + \frac{1}{r} \psi_{-0} - \frac{1}{2r} (f - f_3) \psi_{-0} \right], \\ \lambda_{s1} + \lambda_{t1} &= \frac{i\sqrt{2}}{\phi_1} \left[ \frac{d}{dr} \psi_{+1} - \frac{1}{2r} (f + f_3) \psi_{+1} \right], \\ \lambda_{s1} - \lambda_{t1} &= \frac{i\sqrt{2}}{\phi_2} \left[ \frac{d}{dr} \psi_{-1} - \frac{1}{2r} (f - f_3) \psi_{-1} \right], \end{aligned} \quad (7.11)$$

where

$$\psi_{\pm 0} = \frac{1}{2}(\psi_{s0} \pm \psi_{t0}), \quad \psi_{\pm 1} = \frac{1}{2}(\psi_{s1} \pm \psi_{t1}). \quad (7.12)$$

Dropping the kinetic terms for  $\lambda$ 's in Eqs. (7.2) and substituting (7.11) in these equations we arrive at

$$\begin{aligned} \frac{d}{dr} \psi_{+0} + \frac{1}{r} \psi_{+0} - \frac{1}{2r} (f + f_3) \psi_{+0} - m_L \frac{\phi_1^2}{\xi} \psi_{+1} &= 0, \\ \frac{d}{dr} \psi_{-0} + \frac{1}{r} \psi_{-0} - \frac{1}{2r} (f - f_3) \psi_{-0} - m_L \frac{\phi_2^2}{\xi} \psi_{-1} &= 0, \end{aligned}$$

$$\begin{aligned}
\frac{d}{dr}\psi_{+1} - \frac{1}{2r}(f + f_3)\psi_{+1} - m_L \frac{\phi_1^2}{\xi}\psi_{+0} &= 0, \\
\frac{d}{dr}\psi_{-1} - \frac{1}{2r}(f - f_3)\psi_{-1} - m_L \frac{\phi_2^2}{\xi}\psi_{-0} &= 0,
\end{aligned} \tag{7.13}$$

where  $m_L$  is the light mass given in Eq. (7.10).

Now please observe that the long-range tails of the solutions to these equations are determined by the small mass  $m_L$ , while the string profile functions  $f$  and  $f_3$  are important at much smaller distances  $r \sim 1/g\sqrt{\xi}$  (we assume that both coupling constants are of the same order,  $g_1 \sim g_2 \sim g$ ). This key observation allows us to solve Eqs. (7.13) analytically. We will treat separately two domains,

$$\text{Large } r, \quad r \gg 1/g\sqrt{\xi}, \tag{7.14}$$

$$\text{Intermediate } r, \quad r \lesssim 1/g\sqrt{\xi}. \tag{7.15}$$

In the large- $r$  domain (7.14) we can drop the terms in (7.13) containing  $f$  and  $f_3$  and use the last two equations in (7.13) to express  $\psi_0$  in terms of  $\psi_1$ . We then get

$$\psi_{+0} = \frac{1}{m_L} \frac{d}{dr}\psi_{+1}, \quad \psi_{-0} = \frac{1}{m_L} \frac{d}{dr}\psi_{-1}. \tag{7.16}$$

Substituting this into the first two equations in (7.13) we obtain

$$\begin{aligned}
\frac{d^2}{dr^2}\psi_{+1} + \frac{1}{r} \frac{d}{dr}\psi_{+1} - m_L^2\psi_{+1} &= 0, \\
\frac{d^2}{dr^2}\psi_{-1} + \frac{1}{r} \frac{d}{dr}\psi_{-1} - m_L^2\psi_{-1} &= 0.
\end{aligned} \tag{7.17}$$

These are well-known equations for free fields with mass  $m_L$  in the radial coordinates. Their solutions are<sup>5</sup>

$$\begin{aligned}
\psi_{+1} &= m_L \sqrt{\xi} K_0(m_L r), \\
\psi_{-1} &= m_L \sqrt{\xi} K_0(m_L r),
\end{aligned} \tag{7.18}$$

where  $K_0(x)$  is the imaginary argument Bessel function (the McDonald function). At infinity it falls off exponentially,

$$K_0(x) \sim \frac{e^{-x}}{\sqrt{x}}, \tag{7.19}$$

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<sup>5</sup>Equation (7.17) determines the profile function  $\psi_1$  up to an overall normalization constant. This constant is included in the normalization of the two-dimensional fermion field  $\zeta_R$ . We will discuss this normalization in the next section.

while at  $x \rightarrow 0$  it has the logarithmic behavior,

$$K_0(x) \sim \ln \frac{1}{x}. \quad (7.20)$$

Taking into account (7.16) we get the solutions for the fermion profile functions at  $r \gg 1/m_0$ ,

$$\begin{aligned} \psi_{+1} &\sim m_L \sqrt{\xi} K_0(m_L r), & \psi_{+0} &\sim \sqrt{\xi} \frac{d}{dr} K_0(m_L r), \\ \psi_{-1} &\sim m_L \sqrt{\xi} K_0(m_L r), & \psi_{-0} &\sim \sqrt{\xi} \frac{d}{dr} K_0(m_L r). \end{aligned} \quad (7.21)$$

In particular, at  $1/(g\sqrt{\xi}) \ll r \ll 1/m_L$  we have

$$\psi_{\pm 1} \sim m_L \sqrt{\xi} \ln \frac{1}{m_L r}, \quad \psi_{\pm 0} \sim \sqrt{\xi} \frac{1}{r}. \quad (7.22)$$

In Sect. 8 we will see that the long-range  $1/r$  tails of the profile functions give the leading (logarithmic) contributions to normalization integrals in front of kinetic terms for two-dimensional fermion fields on the string worldsheet (5.2). Therefore, it is essential to determine the precise combination of the profile functions  $\psi_{s0}$  and  $\psi_{t0}$  which has this long-range behavior. To this end we match the behavior at large  $r$  given by (7.21) and (7.22) with the behavior of these functions at intermediate  $r$ , in the domain (7.15).

In this domain we neglect small mass terms in (7.13). We then arrive at

$$\begin{aligned} \frac{d}{dr} \psi_{+0} + \frac{1}{r} \psi_{+0} - \frac{1}{2r} (f + f_3) \psi_{+0} &= 0, \\ \frac{d}{dr} \psi_{-0} + \frac{1}{r} \psi_{-0} - \frac{1}{2r} (f - f_3) \psi_{-0} &= 0, \end{aligned} \quad (7.23)$$

where we restrict ourselves to the equations for the profile functions  $\psi_0$  which have the long-range  $1/r$  tails. These equations are identical to those for the string profile functions, see Eq. (3.2). Therefore, their solutions are known,

$$\psi_{+0} = c_1 \frac{\phi_1}{r}, \quad \psi_{-0} = c_2 \frac{\phi_2}{r}. \quad (7.24)$$

Since the profile function  $\phi_2 \sim \text{const}$  at  $r \rightarrow 0$  the coefficient  $c_2$  above should vanish,

$$c_2 = 0,$$

otherwise the function  $\psi_{-0}$  would be singular at  $r = 0$ . The profile function  $\phi_1 \sim r$  at  $r \rightarrow 0$ . Therefore, finally we get

$$\psi_{s0} = \psi_{t0} = \frac{\phi_1}{r} \quad (7.25)$$

at intermediate  $r$ , and

$$\psi_{s0} = \psi_{t0} = -\sqrt{\xi} \frac{d}{dr} K_0(m_L r) \quad (7.26)$$

at large  $r$ , where we include the constant  $c_1$  in the normalization of the two-dimensional field  $\zeta_R$ , see Sect. 8.

A similar procedure was used in [11] to determine the long-range tails of the superorientational zero modes. It turns out that the profile function  $\psi_-$  (see (7.4)) has the  $1/r$  long-range tail. The result obtained in [11] for this function is

$$\psi_- = \frac{\phi_1}{r\sqrt{\xi}} \quad (7.27)$$

at intermediate  $r$ , and

$$\psi_- = -\frac{d}{dr} K_0(m_L r) \quad (7.28)$$

at large  $r$ .

Note, that the main feature of the fermion zero modes described above is the presence of the long-range tails determined by the *small* mass  $m_L$ . Neither the bosonic string solution (3.7) nor other supertranslational and superorientational fermion zero modes determined by  $\mathcal{N} = 1$  supersymmetry have these  $1/r$  long-range tails. Their presence is the reflection of the Higgs branch which emerges in the bulk theory in the limit  $\mu \rightarrow \infty$  [11].

## 8 Parameters of the heterotic $CP(1)$ model from the bulk theory

To derive kinetic terms for two-dimensional fermions in the string worldsheet theory in the presence of the bulk deformation (2.2) we substitute the fermion zero modes found in Sect. 7 in the fermion kinetic terms of the bulk theory (3.14). As usual we assume the two-dimensional fields to have an adiabatic dependence on the worldsheet coordinates. Note, that the kinetic terms for fields  $S^a$ ,  $x_{0i}$ ,  $\chi_L^a$  and  $\zeta_L$  are not modified. They are still given by Eq. (3.24). The reason is that neither the bosonic string solution (3.7) nor the fermion zero modes associated with unbroken  $\mathcal{N} = 1$  supersymmetry are modified by the deformation (2.2).

Explicitly, the kinetic terms of the worldsheet theory are

$$\begin{aligned} S_{1+1}^{\text{kin}} &= \int d^2x \left\{ 2\pi\xi \left[ \frac{1}{2} (\partial_k x_{0i})^2 + \frac{1}{2} \bar{\zeta}_L i (\partial_0 + i\partial_3) \zeta_L \right. \right. \\ &\quad \left. \left. + \frac{I_\zeta}{2} \bar{\zeta}_R i (\partial_0 - i\partial_3) \zeta_R \right] \right\} \end{aligned}$$

$$\begin{aligned}
& + \beta \left[ \frac{1}{2} (\partial_k S^a)^2 + \frac{I_\chi}{2} \chi_R^a i(\partial_0 - i\partial_3) \chi_R^a + \frac{1}{2} \chi_L^a i(\partial_0 + i\partial_3) \chi_L^a \right. \\
& \left. + I_{\zeta\chi} \chi_R^a \left( i(\partial_0 - i\partial_3) S^a (\zeta_R + \bar{\zeta}_R) + i\varepsilon^{abc} S^b i(\partial_0 - i\partial_3) S^c (\zeta_R - \bar{\zeta}_R) \right) \right] \Big\},
\end{aligned} \tag{8.1}$$

where  $I_\zeta$ ,  $I_\chi$  and  $I_{\zeta\chi}$  are normalization integrals determined by the profile functions of the fermion zero modes. The kinetic terms of  $x_{0i}$  and  $\zeta_L$  are irrelevant; we present them only for completeness. We will consider small and large- $\mu$  limits separately.

## 8.1 Small- $\mu$ limit

To the leading order in  $\mu$

$$I_\zeta = I_\chi = 1, \tag{8.2}$$

i.e the normalization integrals in front of the  $\chi_R^a$  and  $\zeta_R$  kinetic terms are still determined by  $\beta$ . The only new element is the emergence of the bifermion mixing  $\chi_R^a \partial_L S^a \zeta_R$  which is linear in the deformation parameter  $\mu$ . It can be expressed in terms of the profile functions as follows:

$$\begin{aligned}
I_{\zeta\chi} &= \int r dr \left[ -(\lambda_{t1}\lambda_+ + \lambda_{t0}\lambda_-) \frac{\phi_1}{\phi_2} + \frac{g_2^2}{4} (\psi_{t1}\psi_+ + \psi_{t0}\psi_-) \left( 1 + \frac{\phi_1}{\phi_2} \right) \right. \\
& \left. + \frac{g_2^2}{4} (\psi_{s0}\psi_- - \psi_{s1}\psi_+) \left( 1 - \frac{\phi_1}{\phi_2} \right) \right].
\end{aligned} \tag{8.3}$$

Substituting the leading-order fermion profile functions from Eqs. (7.6) – (7.9) in Eq. (8.3) we get

$$I_{\zeta\chi} = -\frac{g^2\mu}{2\sqrt{2}} \int r dr \left[ \frac{g^2(\phi_2^2 - \phi_1^2)^2}{\phi_2^2} \left( 1 + \frac{1}{2}f + \frac{3}{2}f_3 \right) + 4g^2(\phi_1^2 - \phi_2^2)f_3 \right] + O(\mu^2). \tag{8.4}$$

where we put  $g_1 = g_2 = g$ .

Comparing this with Eq. (5.2) we see that the deformation parameter  $\kappa$  is

$$\kappa = I_{\zeta\chi} = \text{const } g^2\mu + O(\mu^2). \tag{8.5}$$

To find the value of the constant in (8.5) one needs to calculate the overlap integral (8.4) numerically. This has not yet been done. What is important is that this constant is just a number, presumably of order one.

We see that in the small- $\mu$  limit the  $\mathcal{N} = (0, 2)$  deformation parameter  $\kappa$  is proportional to  $g^2 \mu$ , in accordance with the Edalati–Tong suggestion. This parameter (of dimension of mass) determines the mass splitting in the  $\mathcal{N} = 2$  multiplets of the bulk theory upon  $\mu$ -deformation, see (2.16).

## 8.2 Large- $\mu$ limit

In this limit the  $\lambda$  fields decouple and the normalization integrals in Eq. (8.1) are given by the  $\psi$ -fermion profile functions,

$$\begin{aligned} I_\zeta &= \frac{1}{\xi} \int r dr \left( \psi_{s0}^2 + \psi_{t0}^2 + \psi_{s1}^2 + \psi_{t1}^2 \right), \\ I_\chi &= 2g_2^2 \int r dr \left( \psi_+^2 + \psi_-^2 \right), \\ I_{\zeta\chi} &= \frac{g_2^2}{2} \int r dr \left( \psi_{t1} \psi_+ + \psi_{t0} \psi_- \right). \end{aligned} \quad (8.6)$$

All three normalization integrals contain large logarithms due to the  $1/r$  long-range tails of the fermion profile functions. Explicitly, substituting here Eqs. (7.26) and (7.28) we get

$$\begin{aligned} I_\zeta &= 2 \ln \frac{m_W}{m_L} + O(1) \\ I_\chi &= 2g_2^2 \left[ \ln \frac{m_W}{m_L} + O(1) \right], \\ I_{\zeta\chi} &= \frac{g_2^2 \sqrt{\xi}}{2} \left[ \ln \frac{m_W}{m_L} + O(1) \right], \end{aligned} \quad (8.7)$$

where  $m_W$  is the gauge field mass (2.12) while  $m_L$  is the small mass (7.10) of the fields  $\tilde{q}$  and their fermionic superpartners. The large logarithmic contributions in (8.7) come from the integration over  $r$  in the domain

$$1/m_W \ll r \ll 1/m_L.$$

Note that the fermion profile functions which have logarithmic behavior in this domain (see Sect. 8.2) do not produce large logarithms. The corrections to the leading behavior in (8.7) come from numerical constants in the arguments of logarithms which we do not control in our approximation.

Now we rescale the fields  $\zeta_R$  and  $\chi_R$  absorbing the logarithms in  $I_\zeta$  and  $I_\chi$  in the normalization of these fields. Then we arrive at the Lagrangian (5.2) with

$$c\kappa = \frac{\kappa}{\sqrt{1 + 8 \frac{\kappa^2}{m_W^2}}} = \frac{m_W}{4}, \quad (8.8)$$



which is equivalent to

$$\begin{aligned}\kappa &= \frac{m_W}{2\sqrt{2}} \left[ 1 + O\left(1/\ln \frac{g_2^2 \mu}{m_W}\right) \right], & c &= \frac{1}{\sqrt{2}} \left[ 1 + O\left(1/\ln \frac{g_2^2 \mu}{m_W}\right) \right], \\ \alpha &= 1 + O\left(1/\ln \frac{g_2^2 \mu}{m_W}\right)\end{aligned}\tag{8.9}$$

at large  $\mu$ . In this regime the worldsheet deformation parameter  $\kappa$  is proportional to the gauge multiplet mass  $m_W$ . Corrections to the leading behavior here go in inverse powers of the large logarithm.

Summarizing, we found the worldsheet deformation parameter from the bulk theory in two limits, small and large  $\mu$ . As was stressed in Sect. 5, the deformation of the worldsheet theory is determined by the single dimensionless parameter  $\alpha$  (see (5.4)), which is the ratio of  $\kappa$  and the gauge boson mass. Our result for this parameter in terms of parameters of the bulk theory reads

$$\alpha = 2\sqrt{2} \frac{\kappa}{m_W} = \begin{cases} \text{const } \frac{g_2^2 \mu}{m_W}, & \text{small } \mu, \\ 1 + O\left(1/\ln \frac{g_2^2 \mu}{m_W}\right), & \text{large } \mu. \end{cases}\tag{8.10}$$

If we translate this behavior in the behavior of the parameter  $\delta$  which enters the gauged formulation of the  $\mathcal{N} = (0, 2)$   $CP(1)$  model (see Sect. 5 and Appendix D for relations between different definitions of the worldsheet deformation parameters) we find that the parameter  $\delta$  tends to infinity in the large- $\mu$  limit. Namely, from (8.10) we get

$$\delta = \frac{\alpha}{\sqrt{1 - |\alpha|^2}} = \begin{cases} \text{const } \frac{g_2^2 \mu}{m_W}, & \text{small } \mu, \\ \text{const } \sqrt{\ln \frac{g_2^2 \mu}{m_W}}, & \text{large } \mu. \end{cases}\tag{8.11}$$

The Edalati–Tong suggestion [13] anticipates  $\delta \sim \mu$  rather than the logarithmic behavior implied by (8.11) at large  $\mu$ .

As was mentioned in Sect. 2 (see also [11]), “large  $\mu$ ” here means the values of  $\mu$  at the upper limit of the window (2.23).

As was already explained, the logarithmic behavior of our results at large  $\mu$  is due to light states with mass  $m_L$ . These states are related to the presence of the Higgs branch in the bulk theory in the limit  $\mu \rightarrow \infty$ .

## 9 Twisted mass in the worldsheet theory

The remainder of this paper is devoted to a more general polynomial bulk theory deformation presented by the superpotential (2.24). Our goal in this section is to

prepare for the analysis of polynomial deformations. Here we will introduce unequal quark mass terms in the undeformed  $\mathcal{N} = 2$  bulk theory and review modifications that occur in the worldsheet theory due to  $m_1 \neq m_2$  [3, 5]. In Sect. 10 we will discuss deformation of the  $\mathcal{N} = 2$  bulk theory by the superpotential (2.24).

Thus, let us drop the assumption (2.3) of equal mass terms for two flavors and introduce a small mass difference  $\Delta m$ . By shifting the adjoint field we can always ensure that

$$m_1 + m_2 = 0. \quad (9.1)$$

Without loss of generality we will assume Eq. (9.1) to be satisfied. With unequal mass terms, the  $U(2)$  gauge group of the bulk theory is broken down to  $U(1)$  by the condensation of the adjoint scalars, see (2.26). The masses of off-diagonal gauge bosons and off-diagonal fields from the squark matrix  $q^{kA}$  (together with their fermion superpartners) get a shift, with splittings proportional to  $\Delta m$  (we assume  $|\Delta m| \ll \sqrt{\xi}$ ).

The Abelian  $Z_2$  strings (3.1) are now *the only* solutions to the first-order string equations. The family of solutions is discrete. The global  $SU(2)_{C+F}$  group is broken down to  $U(1)$  by  $\Delta m \neq 0$ , and the moduli space of the non-Abelian string is lifted. In fact, the vector  $S^a$  gets fixed in two possible positions,  $S^a = (0, 0, \pm 1)$ . If the mass difference is much smaller than  $\sqrt{\xi}$  the set of parameters  $S^a$  becomes *quasimoduli*.

We will outline here derivation of the effective two-dimensional theory on the string worldsheet for unequal mass terms [3]. Under the condition  $\Delta m \neq 0$  we will still be able to introduce orientational quasimoduli  $S^a$ . In terms of the worldsheet model, unequal mass terms lead to a shallow potential for the quasimoduli  $S^a$ . Let us derive this potential.

To this end we start from the expression for the non-Abelian string in the singular gauge (3.7) parametrized by the moduli  $S^a$ , and substitute it in the bulk potential (2.8). The only modification we actually have to make is supplementing our *ansatz* (3.7) by an *ansatz* for the adjoint scalar field  $a^a$ ; the  $U(1)$  scalar field  $a$  will stay fixed at its VEV,  $a = 0$ .

At large  $r$  the field  $a^a$  tends to its VEV aligned along the third axis in the color space,

$$\langle a^3 \rangle = -\frac{\Delta m}{\sqrt{2}}, \quad \Delta m = m_1 - m_2, \quad (9.2)$$

see Eq. (2.26). At the same time, at  $r = 0$  it must be directed along the vector  $S^a$ . The reason for this behavior is easy to understand. The kinetic term for  $a^a$  in Eq. (2.6) contains the commutator term of the adjoint scalar and the gauge potential. The gauge potential is singular at the origin, as is seen from Eq. (3.7). This implies that  $a^a$  must be aligned along  $S^a$  at  $r = 0$ . Otherwise, the string tension would

become divergent. The following *ansatz* for  $a^a$  ensures this behavior:

$$a^a = -\frac{\Delta m}{\sqrt{2}} \left[ \delta^{a3} (1 - \omega) + S^a S^3 \omega \right]. \quad (9.3)$$

Here we introduced a new profile function  $\omega(r)$  which will be determined from a minimization procedure [3]. Note that at  $S^a = (0, 0, \pm 1)$  the field  $a^a$  is given by its VEV, as expected. The boundary conditions for the function  $\omega(r)$  are

$$\omega(\infty) = 0, \quad \omega(0) = 1. \quad (9.4)$$

Substituting Eq. (9.3) in conjunction with (3.7) in the bulk potential (2.8) we get the potential

$$V_{CP(1)} = \beta_{\text{pot}} \int d^2x \frac{|\Delta m|^2}{2} (1 - S_3^2), \quad (9.5)$$

where  $\beta_{\text{pot}}$  is given by the integral

$$\begin{aligned} \beta_{\text{pot}} = & \frac{2\pi}{g_2^2} \int_0^\infty r dr \left\{ \left( \frac{d}{dr} \omega(r) \right)^2 + \frac{1}{r^2} f_3^2 (1 - \omega)^2 \right. \\ & \left. + g_2^2 \left[ \frac{1}{2} \omega^2 (\phi_1^2 + \phi_2^2) + (1 - \omega) (\phi_1 - \phi_2)^2 \right] \right\}. \end{aligned} \quad (9.6)$$

The first and second terms in the integrand come from the kinetic term of the adjoint scalar field  $a^a$  in (2.6) while the term in the square brackets comes from the potential (2.8).

Minimizing with respect to  $\omega(r)$ , with the constraint (9.4), we arrive at

$$\omega(r) = 1 - \frac{\phi_1}{\phi_2}(r). \quad (9.7)$$

This gives

$$\beta_{\text{pot}} = \frac{2\pi}{g_2^2} = \beta. \quad (9.8)$$

We see [3] that the normalization integrals are the same for both, the kinetic and the potential terms in the worldsheet sigma model,  $\beta_{\text{pot}} = \beta$ . As a result we arrive at the following effective theory on the string worldsheet:

$$S_{CP(1)} = \beta \int d^2x \left\{ \frac{1}{2} (\partial_k S^a)^2 + \frac{|\Delta m|^2}{2} (1 - S_3^2) \right\}. \quad (9.9)$$

This is the only functional form that allows  $\mathcal{N} = 2$  completion.<sup>6</sup> For generic  $N$  the potential in the  $CP(N - 1)$  model was obtained in [5].

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<sup>6</sup>Note, that although the global  $SU(2)_{C+F}$  is broken by  $\Delta m$ , the extended  $\mathcal{N} = 2$  supersymmetry is not.

The  $CP(N-1)$  model with the potential (9.9) is nothing but a bosonic truncation of the  $\mathcal{N} = 2$  two-dimensional sigma model which was termed the twisted-mass-deformed  $CP(N-1)$  model. This is a generalization of the massless  $CP(N-1)$  model which preserves four supercharges. Twisted chiral superfields in two dimensions were introduced in [32] while the twisted mass as an expectation value of the twisted chiral multiplet was suggested in [33].  $CP(N-1)$  models with twisted mass were further studied in [34] and, in particular, the BPS spectra in these theories were determined exactly.

The fact that we obtain this form shows that our *ansatz* is fully adequate. The mass-splitting parameter  $\Delta m$  of the bulk theory exactly coincides with the twisted mass of the worldsheet model,

$$m_{\text{tw}} = \Delta m. \quad (9.10)$$

The  $CP(1)$  model (9.9) has two vacua located at  $S^a = (0, 0, \pm 1)$ . Clearly these two vacua correspond to two elementary  $Z_2$  strings.

The twisted-mass-deformed  $CP(N-1)$  model can be written as a strong coupling limit of a  $U(1)$  gauge theory [34], see (3.26). With twisted masses of the  $n^l$  fields taken into account, the bosonic part of the action (3.26) becomes

$$\begin{aligned} S_{CP(1) \text{ bos}} &= \int d^2x \left\{ |\nabla_k n^l|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 + \frac{1}{2e^2} D^2 \right. \\ &\quad \left. + 2 \left| \sigma - \frac{m_l}{\sqrt{2}} \right|^2 |n^l|^2 + iD(|n^l|^2 - 2\beta) \right\}. \end{aligned} \quad (9.11)$$

The vacuum expectation values of the  $\sigma$  field are determined by the quark mass terms. For  $N = 2$  we have

$$\langle \sigma \rangle = \pm \frac{\Delta m}{2\sqrt{2}}. \quad (9.12)$$

In the limit  $e^2 \rightarrow \infty$  the  $\sigma$  field can be eliminated by virtue of an algebraic equation of motion. For  $N = 2$  we get

$$\sigma = \frac{\Delta m}{4\sqrt{2}\beta} (|n^1|^2 - |n^2|^2) = \frac{\Delta m}{2\sqrt{2}} S_3, \quad (9.13)$$

where we also used (3.29). This leads to the potential in (9.9).

## 10 Adding the polynomial deformation superpotential

Now it is time to switch on the polynomial deformation superpotential (2.24). Classically, this deformation does not spoil the BPS nature of the  $Z_2$  strings under consideration. In fact, the string solution remains intact. At  $\Delta m \neq 0$  it is still given

by Eq. (3.1). The reason, as was already mentioned, is that for the deformation superpotential of a special type, with the critical points coinciding with quark mass terms, the fields  $\tilde{q}$  do not condense in the vacuum. The squark VEV's are still given by (2.9). This ensures the field  $\tilde{q}$  to remain unexcited on the string solution (at the classical level). The string is made from the gauge fields and  $q$  fields which have the same mass. Then BPS saturation ensues.

However, the global  $SU(2)_{C+F}$  group is broken by  $\Delta m \neq 0$  already in the undeformed  $\mathcal{N} = 2$  theory, see Sect. 9. As a result  $S^a$  now become quasimoduli and a shallow potential on the moduli space is generated in the effective worldsheet model, see (9.9). Adding the deformation superpotential (2.24) in the bulk theory modifies this potential on the string worldsheet. In this section we will study this modification.

We first focus on modifications of the bosonic potential and then restore the heterotic model in its entirety by calculating the first bosonic term in (4.1).

From (2.8) we extract the deformation of the bosonic potential in the bulk theory

$$\delta V = g^2 \text{Tr} \left| \frac{\partial \mathcal{W}_{3+1}}{\partial \hat{a}} \right|^2, \quad (10.1)$$

where the bulk superpotential is determined by (2.24) and we take  $g_1 = g_2 = g$  to simplify calculations.

As for the adjoint field  $a^a$  we take the same *ansatz* (9.3) as was used in Sect. 9 for unequal quark mass terms. Substituting it in (10.1) we get

$$\delta V_{1+1} = g^2 \frac{\pi}{32} |\mu \Delta m|^2 \int d^2 x \left(1 - S_3^2\right)^2 \int r dr \omega^2 (2 - \omega)^2. \quad (10.2)$$

The profile function  $\omega$  here should be determined via minimization procedure, much in the same way as it was done in Sect. 9 for  $\mathcal{N} = 2$  theory. We consider separately the cases of small and large  $\mu$ . The fact that it is the square of  $1 - S_3^2$  that enters tells us that supersymmetry of the worldsheet model cannot be  $\mathcal{N} = (2, 2)$ . Then, it must be  $\mathcal{N} = (0, 2)$ .

## 10.1 Small- $\mu$ limit

For small  $\mu$  we consider (10.2) as a perturbation. To the leading order in  $\mu$  we use the expression (9.7) for the profile function  $\omega$  obtained in the  $\mathcal{N} = 2$  limit. Then the deformation of the worldsheet theory is

$$\delta V_{1+1} = \beta \frac{I}{64} \frac{g^4 |\mu|^2 |\Delta m|^2}{m_W^2} \int d^2 x \left(1 - S_3^2\right)^2, \quad (10.3)$$

where  $I$  is a dimensionless numerical factor determined by the string profile functions

$$I = m_W^2 \int r dr \left( 1 - \frac{\phi_1^2}{\phi_2^2} \right)^2. \quad (10.4)$$

The bosonic part of the effective theory on the string worldsheet is given by the sum of the twisted mass  $CP(1)$  model, Eq. (9.9), and the deformation potential (10.3). We see that the points  $S^a = (0, 0, \pm 1)$  remain to be the vacua of the deformed theory. The corresponding vacuum energy density vanishes at the classical level. These vacua describe two  $Z_2$  strings of the bulk theory.  $\mathcal{N} = (0, 2)$  supersymmetry is not broken at the classical level.

Now, we can rewrite the potential (10.3) in the form of the deformation of the  $CP(1)$  model (9.11) in the gauged formulation, see Eq. (4.1). To the leading order in  $\mu$  the  $\sigma$  field is determined by Eq. (3.32) obtained in the  $\mathcal{N} = (2, 2)$  limit. Therefore we can write (10.3) as

$$\delta V_{1+1} = \int d^2x \frac{8\beta}{m_W^2} \left| \frac{\partial \mathcal{W}_{1+1}}{\partial \sigma} \right|^2 \quad (10.5)$$

with

$$\frac{\partial \mathcal{W}_{1+1}}{\partial \sigma} = \frac{\sqrt{I}}{2\sqrt{2}} \frac{g^2 \mu}{\Delta m} \left( \sigma^2 - \frac{\Delta m^2}{8} \right). \quad (10.6)$$

This result is in accordance with the Edalati–Tong suggestion [13], see Sect. 4. The critical points of the two-dimensional superpotential are determined by the quark mass terms, so supersymmetry is not broken in the vacua (9.12) at the classical level. From the standpoint of the bulk theory this means that  $Z_2$  strings are BPS saturated. This condition was the motivation behind the Edalati–Tong suggestion [13]. The coefficient in front of the polynomial in  $\sigma$  in  $\mathcal{W}_{1+1}$  is proportional to  $g^2 \mu$ , much in the same way as for deformation (2.2), see Eq. (8.5). This parameter determines the mass splitting in  $\mathcal{N} = 2$  multiplets of the bulk theory in the small- $\mu$  limit, see Sect. 2.

## 10.2 Large- $\mu$ limit

In this limit we have to add the deformation (10.2) to the  $CP(1)$  model potential (9.5) and carry out minimization in order to find the modified profile function  $\omega(r)$ .

If  $S_3^2$  is very close to unity,

$$1 - S_3^2 \ll m_W / g^2 \mu,$$

then the deformation in (10.2) still can be considered as a perturbation, much in the same way as it was done Sect. 10.1. In this case the profile function  $\omega(r)$  stays intact and the result for deformation of the worldsheet theory is still given by Eqs. (10.3)

and (10.6). However, let us consider the range of  $S_3^2$  not too close to unity. In this case the deformation (10.2) becomes large at large  $\mu$ . Minimization with respect to  $\omega(r)$  requires  $\omega$  to tend to zero in this limit. However, the boundary conditions (9.4) tell us that  $\omega(r)$  cannot vanish for all  $r$ .

Taking this into account it is natural to assume the following simple profile for  $\omega(r)$ :

$$\omega = \begin{cases} 1, & r < r_0, \\ 0 & r > r_0, \end{cases} \quad (10.7)$$

where the parameter  $r_0$  should be found from minimization. It will turn out to be very close to zero.

Indeed,

$$V_{1+1} = \beta \int d^2x \left\{ \frac{|\Delta m|^2}{2} (1 - S_3^2) \ln \frac{1}{r_0 m_W} + \frac{g^4}{64} |\mu \Delta m|^2 (1 - S_3^2)^2 \frac{r_0^2}{2} \right\}, \quad (10.8)$$

where the leading logarithmic contribution comes from the second term in (9.6). The upper limit of the logarithmic integral over  $r$  is given by the inverse gauge boson mass. At  $r \sim 1/m_W$  the profile function  $f_3$  is no longer constant. It cuts the logarithmic integration. Minimizing with respect to  $r_0$  we obtain

$$r_0^2 \sim \frac{1}{g^4 |\mu|^2 (1 - S_3^2)}. \quad (10.9)$$

We see that  $r_0$  is very close to zero, indeed. It is determined by the mass of the adjoint fields  $g^2 \mu$  which tends to infinity at  $\mu \rightarrow \infty$ .

Substituting this back in Eq. (10.8) we get

$$V_{1+1} = \beta \int d^2x \frac{|\Delta m|^2}{2} (1 - S_3^2) \left[ \ln \frac{g^2 |\mu|}{m_W} + \frac{1}{2} \ln (1 - S_3^2) \right]. \quad (10.10)$$

This is our final result for the bosonic potential in the  $\mathcal{N} = (0, 2)$  worldsheet theory on the string. Undetermined non-logarithmic corrections to this leading logarithmic expression come from the second term in (10.8), other terms in (9.6), as well as from improvements of the simple step-function profile (10.7).

If  $S_3^2$  is not too close to unity, we can neglect the second term in (10.10). We see that the main effect of the bulk deformation is the modification of the twisted mass parameter of the  $CP(1)$  model. Now, expressed in terms of the bulk parameters it acquires a dependence on  $\mu$  and  $\xi$ . Instead of the simple expression (9.10) we now get an “amplified” twisted mass,

$$m_{\text{tw}} = \Delta m \sqrt{\ln \frac{g^2 |\mu|}{m_W}}. \quad (10.11)$$

The twisted mass becomes logarithmically large as we increase  $\mu$ . The worldsheet theory coupling becomes weaker.

If  $S_3^2$  is close to unity,  $m_W/g^2\mu \ll 1 - S_3^2 \ll 1$  we keep the second term in (10.10) as a correction to the leading twisted mass term. This term also has zeros at  $S^a = (0, 0, \pm 1)$ . Therefore, the vacua of the worldsheet theory remain intact and supersymmetry is not broken at the classical level. However, the potential becomes nonpolynomial in  $S_3$ .

We can rewrite our results in terms of the gauged formulation. We have

$$\begin{aligned}
S_{1+1} = & \int d^2x \left\{ |\nabla_k n^l|^2 + \frac{1}{4e^2} F_{kl}^2 + \frac{1}{e^2} |\partial_k \sigma|^2 + \frac{1}{2e^2} D^2 \right. \\
& + 2 \left| \sigma - \frac{m_{\text{tw}}}{2\sqrt{2}} \right|^2 |n^1|^2 + 2 \left| \sigma + \frac{m_{\text{tw}}}{2\sqrt{2}} \right|^2 |n^2|^2 + iD(|n^l|^2 - 2\beta) \\
& \left. + \beta \frac{|\Delta m|^2}{4} \left| \left( 1 - \frac{8\sigma^2}{m_{\text{tw}}^2} \right) \ln \left( 1 - \frac{8\sigma^2}{m_{\text{tw}}^2} \right) \right| \right\}, \tag{10.12}
\end{aligned}$$

where the twisted mass is given by (10.11). The last term in the potential here is small and can be considered as a perturbation. Then the equation of motion for the  $\sigma$  field gives, to the leading order,

$$\sigma = \frac{m_{\text{tw}}}{4\sqrt{2}\beta} (|n^1|^2 - |n^2|^2) = \frac{m_{\text{tw}}}{2\sqrt{2}} S_3. \tag{10.13}$$

Being substituted in the last term in (10.12) it reproduces the last logarithmic potential term in (10.10).

The last term in (10.12) can be written in the form (10.5) with the  $\mathcal{N} = (0, 2)$  superpotential

$$\frac{\partial \mathcal{W}_{1+1}}{\partial \sigma} = m_W \frac{\Delta m}{4\sqrt{2}} \left[ \left( 1 - \frac{8\sigma^2}{m_{\text{tw}}^2} \right) \ln \left( 1 - \frac{8\sigma^2}{m_{\text{tw}}^2} \right) \right]^{1/2}. \tag{10.14}$$

This superpotential is non-polynomial and does not satisfy conjecture (4.3). It depends on  $\mu$  logarithmically via the twisted mass (10.11). Its critical points are determined by the twisted mass and coincide with the vacua of the theory (10.12)

$$\langle \sigma \rangle = \pm \frac{m_{\text{tw}}}{2\sqrt{2}}. \tag{10.15}$$

The vacua of the deformed theory are modified as compared with the  $\mathcal{N} = (2, 2)$  case, see Eq. (9.12).

We would like to stress the following: the most important impact of the  $\mathcal{N} = 2$  breaking polynomial deformation of the bulk theory at large  $\mu$  is the logarithmic



dependence of the worldsheet twisted mass (10.11) on the ratio  $\mu/\sqrt{\xi}$ . In the  $\mathcal{N} = 2$  limit the dependence of  $m_{\text{tw}}$  on the nonholomorphic FI parameter  $\xi$  is forbidden [3, 5]. This is no longer true as we break  $\mathcal{N} = 2$  supersymmetry down to  $\mathcal{N} = 1$  in the bulk. As a result the twisted mass term becomes large forcing the string orientational vector  $S^a$  to point towards the north or south poles of  $S_2 = \text{SU}(2)/\text{U}(1)$ . This means that the string becomes more “Abelian” as we increase  $\mu$ . This is in accord with “Abelianization” of the bulk theory. As was mentioned in Sect. 2, the  $\mu$  deformation splits adjoint scalar multiplet giving large masses to  $a$  and  $a^3$  components while the masses of the  $a^{1,2}$  components are still determined by  $m_W$  (and  $\Delta m$ ).

Once  $S_3^2$  is close to unity this effect is partly washed out in the worldsheet theory by the logarithmic correction in (10.10).

Note that small variations of the polynomial deformation (shifting the critical points from  $m_{1,2}$ ) ruin the BPS saturation of the classical string solutions. The worldsheet model in this case must break supersymmetry already at the classical level. This means that there is no protection against spontaneous supersymmetry breaking even in the case when the critical points coincide with  $m_{1,2}$ . Classically SUSY is unbroken, the breaking can (and does) occur at the quantum level.

## 11 Conclusions

In this paper we continue studies of non-Abelian strings in the  $\text{U}(N)$  bulk theories with  $N$  flavors. If the bulk theory is  $\mathcal{N} = 2$  supersymmetric, the string is BPS saturated and the low-energy theory on the string worldsheet has  $\mathcal{N} = (2, 2)$  supersymmetry. The worldsheet model splits into two completely disconnected sectors: (i) noninteracting theory of two translational and four supertranslational moduli; and (ii) supersymmetric  $CP(N-1)$  model describing interactions of orientational and superorientational moduli.

We start from deforming the  $\mathcal{N} = 2$  bulk theory by introducing deformations (of a special type), which preserve  $\mathcal{N} = 1$  in the bulk. The string solution at the classical level remains BPS saturated. Normally, this would imply conservation of two supercharges on the string worldsheet. Previously it was believed, however, that worldsheet supersymmetry gets an “accidental” enhancement. This is due to the facts that  $\mathcal{N} = (1, 1)$  SUSY is automatically elevated up to  $\mathcal{N} = (2, 2)$  on  $CP(N-1)$  and, at the same time, there are no “heterotic”  $\mathcal{N} = (0, 2)$  generalizations of the bosonic  $CP(N-1)$  model.

Edalati and Tong noted that the target space is in fact  $CP(N-1) \times C$  rather than  $CP(N-1)$ . If two fermionic moduli from the first sector (see above) become coupled to moduli from the second sector, one can build a heterotic  $\mathcal{N} = (0, 2)$  model

with the  $CP(N-1)$  target space for the bosonic moduli. They suggested a general structure of such model (in the gauged formulation), and a particular formula relating deformation parameters in the bulk with those on the worldsheet. Later Tong argued that  $\mathcal{N} = (0, 2)$  supersymmetry of the heterotic model is spontaneously broken at the quantum level.

Our task was a direct derivation of the string worldsheet model from the bulk theory with  $\mathcal{N} = 1$  and the superpotential (2.2) or (2.24), including the relation between the deformation parameters in the bulk and on the worldsheet. The model we obtain follows the general pattern of Edalati and Tong. Both, the  $O(3)$  formulation and the geometric formulation which we use in this paper instead of the gauged formulation serve well our original goal and allowed us to find the full solution. The Edalati–Tong suggestion as to how the bulk and worldsheet deformation parameters must be related to each other turns out to be true only for small values of  $\mu$ . For large deformations our expressions are different.

As was mentioned more than once, deformation of the worldsheet theory is determined by a single *dimensionless* parameter  $\alpha$  (or  $\gamma$ , see (5.10)). Our result for this parameter is given in Eq. (8.10). The only dimensional (mass) parameter of the worldsheet theory  $\Lambda_{CP(1)}$  is generated by nonperturbative effects in two dimensions. In particular, the  $\mathcal{N} = (0, 2)$  deformation does not involve any new mass parameters.

We derived the heterotic  $\mathcal{N} = (0, 2)$  model with the  $CP(N-1)$  target space for bosonic fields in the geometric formulation (see Eq. (6.39)). This representation turns out to be very convenient for proving spontaneous breaking of SUSY at small  $\mu$ . The vacuum energy density is shown to be proportional to the square of the bifermion condensate. Spontaneous breaking for arbitrary values of deformation parameter (and large  $N$ ) will be proven in [17].

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# Appendices

## A. Euclidean notation

As was mentioned, in Sects. 2-5 and 7-11 we use a formally Euclidean notations, e.g.

$$F_{\mu\nu}^2 = 2F_{0i}^2 + F_{ij}^2, \quad (\text{A.1})$$

and

$$(\partial_\mu a)^2 = (\partial_0 a)^2 + (\partial_i a)^2, \quad (\text{A.2})$$

etc. This is appropriate, since we mostly consider static (time-independent) field configurations, and  $A_0 = 0$ . Then the Euclidean action is nothing but the energy functional.

Then, in the fermion sector we have to define the Euclidean matrices

$$(\sigma_\mu)^{\alpha\dot{\alpha}} = (1, -i\vec{\tau})_{\alpha\dot{\alpha}}, \quad (\text{A.3})$$

and

$$(\bar{\sigma}_\mu)_{\dot{\alpha}\alpha} = (1, i\vec{\tau})_{\dot{\alpha}\alpha}. \quad (\text{A.4})$$

Lowering and raising of the spinor indices is performed by virtue of the antisymmetric tensor defined as

$$\begin{aligned} \varepsilon_{12} &= \varepsilon_{\dot{1}\dot{2}} = 1, \\ \varepsilon^{12} &= \varepsilon^{\dot{1}\dot{2}} = -1. \end{aligned} \quad (\text{A.5})$$

The same raising and lowering convention applies to the flavor  $\text{SU}(2)_R$  indices  $f, g$ , etc.

When the contraction of the spinor indices is assumed inside the parentheses we use the following notation:

$$(\lambda\psi) \equiv \lambda_\alpha \psi^\alpha, \quad (\bar{\lambda}\bar{\psi}) \equiv \bar{\lambda}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}}. \quad (\text{A.6})$$

The bar (overline) denotes Hermitian conjugation both in four and two dimensions.

## B. Two-dimensional Minkowski notation

$$\gamma^0 = \gamma^t = \sigma_2, \quad \gamma^1 = \gamma^z = i\sigma_1, \quad \gamma^0\gamma^1 = \sigma_3. \quad (\text{B.1})$$

$$g^{\mu\nu} = \text{diag}\{+1, -1\}. \quad (\text{B.2})$$

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \quad \bar{\psi} = \psi^\dagger \gamma^0, \quad \bar{\theta} = \theta^\dagger \gamma^0. \quad (\text{B.3})$$

$$\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \quad \partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}. \quad (\text{B.4})$$

## C. Witten index

Witten was the first to calculate  $\text{Tr}(-1)^F$  in  $CP(N-1)$  models, see Sect. 10 of [16]. It turns out to coincide with the Euler characteristic of the target space, i.e.  $I_W = N$  for  $CP(N-1)$ . For  $CP(1)$  treated in a small box, as in [16], we have two vacua (e.g. the north and south poles of the sphere), both of them bosonic. Hence  $I_W = 2$ . Introduction of an extra field  $\zeta_R$ , as in Eq. (6.25), splits each of these vacua in two, one bosonic and one fermionic (since the  $\zeta_R$  zero level can be either filled or empty). Thus, in the heterotic  $\mathcal{N} = (0, 2)$  model based on  $CP(1)$

$$I_W = 0. \quad (\text{C.1})$$

This is in full agreement with the consideration carried out in Sect. 6 where it was proven that supersymmetry is spontaneously broken at small but nonvanishing values of  $\gamma$ . A proof for generic values of the deformation parameter but large  $N$  is presented in [17].

## D. Worldsheet deformation parameters

For convenience in this Appendix we summarize different definitions of the worldsheet deformation parameter.

In  $O(3)$  sigma model formulation given in Sect. 5  $\mathcal{N} = (0, 2)$  deformation parameter  $\alpha$  is related to parameters which enter the action (5.2) as

$$\begin{aligned} \alpha &\equiv \frac{2\sqrt{2}\kappa}{m_W} = \frac{2\sqrt{2}\kappa}{g_2\sqrt{\xi}}, \\ c^2 &= \frac{1}{1 + |\alpha|^2}. \end{aligned} \quad (\text{D.1})$$

Its relation to the deformation parameter  $\gamma$  which is used for the geometric formulation of  $\mathcal{N} = (0, 2)$   $CP(1)$  model in Sect. 6 is the following:

$$\gamma = \sqrt{\beta} \frac{\alpha}{\sqrt{1 + |\alpha|^2}}. \quad (\text{D.2})$$

In the gauged formulation of the model (see Sect. 4) it is convenient to use another parameter  $\delta$ ,

$$\delta = \frac{\alpha}{\sqrt{1 - |\alpha|^2}}. \quad (\text{D.3})$$

In the limit of small and large  $\mu$  we have

$$\alpha = 2\sqrt{2} \frac{\kappa}{m_W} = \begin{cases} \text{const } \frac{g^2\mu}{m_W}, & \text{small } \mu, \\ 1 + O(1/\ln \frac{g^2\mu}{m_W}), & \text{large } \mu, \end{cases} \quad (\text{D.4})$$

and

$$\delta = \frac{\alpha}{\sqrt{1 - |\alpha|^2}} = \begin{cases} \text{const } \frac{g^2 \mu}{m_W}, & \text{small } \mu, \\ \text{const } \sqrt{\ln \frac{g_2^2 \mu}{m_W}}, & \text{large } \mu. \end{cases} \quad (\text{D.5})$$

## References

- [1] A. Hanany and D. Tong, JHEP **0307**, 037 (2003) [hep-th/0306150].
- [2] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, Nucl. Phys. B **673**, 187 (2003) [hep-th/0307287].
- [3] M. Shifman and A. Yung, Phys. Rev. D **70**, 045004 (2004) [hep-th/0403149].
- [4] D. Tong, Phys. Rev. D **69**, 065003 (2004) [hep-th/0307302].
- [5] A. Hanany and D. Tong, JHEP **0404**, 066 (2004) [hep-th/0403158].
- [6] D. Tong, *TASI Lectures on Solitons*, arXiv:hep-th/0509216.
- [7] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, J. Phys. A **39**, R315 (2006) [arXiv:hep-th/0602170].
- [8] M. Shifman and A. Yung, *Supersymmetric Solitons*, Rev. Mod. Phys. **79** 1139 (2007) [arXiv:hep-th/0703267].
- [9] J. Wess and J. Bagger, *Supersymmetry and supergravity*, (Princeton Univ. Press, Second Ed., 1992).
- [10] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Phys. Rept. **116**, 103 (1984).
- [11] M. Shifman and A. Yung, Phys. Rev. D **72**, 085017 (2005) [hep-th/0501211].
- [12] A. Ritz, M. Shifman and A. Vainshtein, Phys. Rev. D **70**, 095003 (2004) [arXiv:hep-th/0405175].
- [13] M. Edalati and D. Tong, JHEP **0705**, 005 (2007) [arXiv:hep-th/0703045].
- [14] E. Witten, *Two-dimensional models with (0,2) supersymmetry: Perturbative aspects*, arXiv:hep-th/0504078.
- [15] A. I. Vainshtein and A. Yung, Nucl. Phys. B **614**, 3 (2001) [hep-th/0012250].
- [16] E. Witten, Nucl. Phys. B **202**, 253 (1982). [Reprinted in *Supersymmetry*, Ed. S. Ferrara (North/Holland/World Scientific, 1987), Vol. 1, p. 490].
- [17] M. Shifman and A. Yung, in preparation.
- [18] D. Tong, JHEP **0709**, 022 (2007) [arXiv:hep-th/0703235].

- [19] A. Hanany, M. J. Strassler and A. Zaffaroni, Nucl. Phys. B **513**, 87 (1998) [hep-th/9707244].
- [20] K. Intriligator and N. Seiberg, Nucl. Phys. (Proc. Suppl.) **45BC**, 1 (1996) [hep-th/9509066].
- [21] A. Abrikosov, Sov. Phys. JETP **32** 1442 (1957) [Reprinted in *Solitons and Particles*, Eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 356];  
H. Nielsen and P. Olesen, Nucl. Phys. **B61** 45 (1973) [Reprinted in *Solitons and Particles*, Eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984), p. 365].
- [22] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **71**, 045010 (2005) [hep-th/0412082].
- [23] A. M. Polyakov, Phys. Lett. B **59**, 79 (1975).
- [24] E. Witten, Phys. Rev. D **16**, 2991 (1977).
- [25] P. Di Vecchia and S. Ferrara, Nucl. Phys. B **130**, 93 (1977).
- [26] E. Witten, Nucl. Phys. B **403**, 159 (1993) [hep-th/9301042].
- [27] A. Gorsky, M. Shifman and A. Yung, Phys. Rev. D **75**, 065032 (2007) [hep-th/0701040].
- [28] A. Ritz, private communication (2006).
- [29] B. Zumino, Phys. Lett. B **87**, 203 (1979).
- [30] D. Kutasov, A. Schwimmer and N. Seiberg, Nucl. Phys. B **459**, 455 (1996) [hep-th/0510222].
- [31] A. Gorsky, A. I. Vainshtein and A. Yung, Nucl. Phys. B **584** (2000) 197 [hep-th/0004087].
- [32] L. Alvarez-Gaume and D. Z. Freedman, Commun. Math. Phys. **91**, 87 (1983);  
S. J. Gates, Nucl. Phys. B **238**, 349 (1984); S. J. Gates, C. M. Hull and M. Roček,  
Nucl. Phys. B **248**, 157 (1984).
- [33] A. Hanany and K. Hori, Nucl. Phys. B **513**, 119 (1998) [hep-th/9707192].
- [34] N. Dorey, JHEP **9811**, 005 (1998) [hep-th/9806056].